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UNIVERSITY OF ALBERTA

STUDIES IN BOX SPLINES

by

Natarajan Sivakumar

A THESIS SUBMITTED TO
THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA
SPRING, 1990

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S.D. Riemenschneider

Z. Ditzian

A. Meir

R.E. Rink

C.K. Chui

Date: April 20, 1990

TO MY PARENTS, NATARAJAN AND KAMESWARI,

AND

MY TEACHERS OF YESTERDAY, TODAY, AND TOMORROW.

ABSTRACT

This thesis deals with two basic aspects of box spline theory; cardinal interpolation by integer translates of the box spline, and algebraic properties associated with box spline spaces.

The first chapter of the thesis provides a general introduction and some background material. Chapter 2 uses Fourier analytic methods to recover a known result characterizing the correctness of cardinal interpolation by shifted univariate B–splines. A convergence theorem for such splines is also established therein. The third chapter examines the feasibility of cardinal interpolation by shifted bivariate box splines on a three-direction mesh. A correctness result is obtained for a specific subclass of such splines.

Chapters 4 and 5 are devoted to the study of certain algebraic features of complex exponential box splines associated with rational matrices. Precisely, a sufficient condition for the linear independence of integer translates of such box splines is derived in Chapter 4. A sufficient condition is found in Chapter 5 for the finite dimensionality of the space of sequences which give rise to linear dependence of box spline translates. These results extend the corresponding theorems known for box splines associated with integer matrices.

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It was my privilege to have been a student and colloborator of Professor Rong Qing Jia during the academic year 1988–89. His remarkable mathematical gifts and indefatigable energy were a constant source of inspiration and a major portion of this thesis owes its existence to the ideas he imparted.

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. Preamble

Let ψ be a compactly supported distribution on the s-dimensional real Euclidean space $\mathbf{R}^s$. Over the years, one of the central themes of study in the theory of approximation has been the analysis of the space

$$S(\psi) := \left\{ \sum_{j \in \mathbf{Z}^s} a(j)\psi(\cdot - j) : \ a : \mathbf{Z}^s \to \mathbf{C} \right\} \tag{1.1.1}$$

of (multi-)integer translates of ψ. This analysis can be broadly classified into four categories.

(a) Algebraic properties connected with $S(\psi)$;

(b) Function classes representable in $S(\psi)$ (i.e., polynomials, entire functions etc., that are expressible as integer translates of ψ);

(c) Approximation power of $S(\psi)$ (i.e., the extent to which given functions can be approximated by elements of $S(\psi)$);

(d) Cardinal interpolation from $S(\psi)$ (i.e., seeking elements from $S(\psi)$ that interpolate prescribed data on the integer lattice).

This thesis makes some modest contributions towards certain questions that arise in the context of categories (a) and (d).

Several general results, in each of the genres listed above, are available in the literature. Apart from their generality, such results also serve to shed a good

deal of light when one studies particular distributions about whose structure and properties more precise information is available.

In the past decade, an especially interesting and popular choice of ψ has been the multivariate **box spline**. In order to recall its definition, we follow a reverse chronological order and begin with the **exponential box spline**.

Let Ξ be an $s \times n$ real matrix with non-zero columns $x_1, x_2, \ldots, x_n$ and let $\lambda \in \mathbf{C}^n$. The centred exponential box spline $B_{\Xi,\lambda}$ associated with Ξ and λ is the distribution on $\mathbf{R}^s$ given by the rule

$$\langle B_{\Xi,\lambda}, f \rangle := \int_{[-\frac{1}{2},\frac{1}{2}]^n} e^{\lambda \cdot t} f(\Xi t)\, dt, \qquad f \in C(\mathbf{R}^s). \tag{1.1.2}$$

Equivalently, $B_{\Xi,\lambda}$ is also given by its Fourier transform

$$\widehat{B}_{\Xi,\lambda}(y) := \prod_{k=1}^{n} \frac{\sin((i\lambda_k + x_k \cdot y)/2)}{(i\lambda_k + x_k \cdot y)/2}, \qquad y \in \mathbf{C}^s. \tag{1.1.3}$$

Indeed, this particularly simple form of its Fourier transform is one of the more attractive features of the box spline. In general, $B_{\Xi,\lambda}$ is a compactly supported distribution whose support is contained in the set

$$\left\{ \sum_{i=1}^{n} t_i x_i : -1/2 \le t_i \le 1/2 \right\}.$$

However, if rank $\Xi = s$, then $B_{\Xi,\lambda}$ is a piecewise exponential polynomial function (thus the name exponential box spline) on $\mathbf{R}^s$.

The most important special case of $B_{\Xi,\lambda}$ is the case $\lambda = 0$. This corresponds to the (centred) **polynomial box spline** B_Ξ introduced earlier by de Boor and DeVore [BD83] as the exact multi-dimensional analogue of the

well-known univariate B-spline of Schoenberg's. Once again, B_Ξ is a compactly supported distribution in general. If rank $\Xi = s$, then it is a piecewise polynomial on $\mathbf{R}^s$ (hence the term spline) of total degree at most $n - s$. Its Fourier transform is obtained by setting $\lambda = 0$ in (1.1.3):

$$\widehat{B}_\Xi(y) := \prod_{k=1}^{n} \frac{\sin((x_k \cdot y)/2)}{(x_k \cdot y)/2}, \qquad y \in \mathbf{R}^s. \tag{1.1.4}$$

Several basic properties of polynomial box splines were established by de Boor and Höllig [BH82] and many others were discovered by Dahmen and Micchelli [DM85].

Exponential box splines were initially introduced and studied for $\lambda \in \mathbf{R}^n$ (these are called real exponential box splines) by Amos Ron [R88]. Extensive analysis of their complex counterparts (i.e., $\lambda \in \mathbf{C}^n$) followed with the works of Ron [R86], Ben-Artzi and Ron [BAR88], and Dahmen and Micchelli [DM89]. The apparent simplicity of Definition (1.1.2) belies the remarkable richness of structure possessed by the box splines; so much so that their theory has developed at a rapid pace and continues to flourish. Its interaction with other areas of mathematics is being explored actively and the existent theory has already lent itself to several applications in other disciplines.

1.2. Selected examples

Next we discuss three special examples of polynomial box splines. The first two of these will be encountered again in the next two chapters of this thesis while some aspects of the third will provide the motivation for our study in Chapter 4.

Example 1.2.1. Ξ is the $1 \times n$ matrix given by

$$\Xi = (1 \quad 1 \quad \dots \quad 1). \tag{1.2.2}$$

Discussion. This is the (centred) univariate B-spline of degree $n-1$ and will be denoted by B_n. From (1.1.4), we have

$$\widehat{B}_n(t) = [S(t)]^n \tag{1.2.3}$$

where

$$S(t) := \frac{\sin(t/2)}{(t/2)}. \tag{1.2.4}$$

♠

Example 1.2.5. Ξ is the $2 \times n$ matrix given by

$$\Xi = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & 1 & \dots & 1 \\ 0 & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 \end{pmatrix}. \tag{1.2.6}$$

Discussion. The vectors $(1,0)$, $(0,1)$, and $(1,1)$ are assumed to occur with multiplicities k, l, and m, respectively (with $k+l+m=n$). The matrix (1.2.6) represents the **three-direction mesh** on the plane. The corresponding box spline will be denoted by $B_{k,l,m}$ and referred to as the bivariate box spline on the three-direction mesh. Once again, (1.1.4) implies

$$\widehat{B}_{k,l,m}(t_1,t_2) = [S(t_1)]^k\,[S(t_2)]^l\,[S(t_1+t_2)]^m, \quad (t_1,t_2) \in \mathbf{R}^2, \tag{1.2.7}$$

where S is the sinc function defined by (1.2.4).

♠

Example 1.2.8. Ξ is the $2 \times n$ matrix given by

$$\Xi = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 \\ 0 & \dots & 0 & 1 & \dots & 1 & 1 & \dots & 1 & -1 & \dots & -1 \end{pmatrix}. \tag{1.2.9}$$

Discussion. Assume that the vectors $(1,0)$, $(0,1)$, $(1,1)$, and $(1,-1)$ repeat themselves with multiplicities k, l, m, and p, respectively, with $k+l+m+p=n$. The matrix Ξ given above represents the **four-direction mesh** on the plane and the box spline B_Ξ associated with it is called the bivariate box spline on the four-direction mesh. The latter will be denoted by $B_{k,l,m,p}$ and its Fourier transform, by virtue of (1.1.4), is given by

$$\widehat{B}_{k,l,m,p}(t_1,t_2) = [S(t_1)]^k\,[S(t_2)]^l\,[S(t_1+t_2)]^m\,[S(t_1-t_2)]^p\,, \quad (t_1,t_2) \in \mathbf{R}^2. \tag{1.2.10}$$

♠

1.3. Cardinal Interpolation

We now embark on a brief discussion of the cardinal interpolation problem which will constitute our mainstay in the first half of this thesis. Suppose that ψ is a bounded compactly supported function defined on $\mathbf{R}^s$ and recall that

$$S(\psi) := \left\{ \sum_{j \in \mathbf{Z}^s} a(j)\psi(\cdot - j) : \ a : \mathbf{Z}^s \to \mathbf{C} \right\}.$$

The problem of **cardinal interpolation** by ψ of prescribed data $f : \mathbf{Z}^s \to \mathbf{C}$ is to find a sequence $a_f : \mathbf{Z}^s \to \mathbf{C}$ such that

$$\sum_{j \in \mathbf{Z}^s} a_f(j)\psi(k-j) = f(k) \qquad \forall k \in \mathbf{Z}^s. \tag{1.3.1}$$

Cardinal interpolation by ψ is said to be **correct** if for every bounded data sequence $f : \mathbf{Z}^s \to \mathbf{C}$, there exists a unique bounded sequence a_f such that (1.3.1) holds. Contrariwise, cardinal interpolation with ψ is called **singular** is there exists a non-trivial bounded sequence a such that

$$\sum_{j \in \mathbf{Z}^s} a(j)\psi(k-j) = 0 \quad \forall k \in \mathbf{Z}^s. \tag{1.3.2}$$

Associated with the problem of cardinal interpolation by ψ is the **characteristic polynomial** P_ψ defined by

$$P_\psi(x) := \sum_{j \in \mathbf{Z}^s} \psi(j)e^{-ijx}, \qquad x \in \mathbf{R}^s. \tag{1.3.3}$$

The reason for and the manner of this polynomial's entry into the picture is expounded well in Riemenschneider's recent survey article [Rie89]. For our part, we shall rest content by merely documenting the following crucial connexion that exists between the cardinal interpolation problem and the associated characteristic polynomial.

Theorem 1.3.4. *Let ψ be a bounded compactly supported function defined on $\mathbf{R}^s$. Then cardinal interpolation by ψ is correct if and only if P_ψ does not vanish on $\mathbf{R}^s$.*

A proof of this theorem can be found in de Boor, Höllig, and Riemenschneider [BHR85] and Chui, Jetter, and Ward [CJW87]. Further implications of the non-vanishing of P_ψ, especially concerning the nature of the interpolant, are also indicated in these sources as well as in [Rie89]. (See also [C88].)

It is our intent, in this thesis, to apply Theorem 1.3.4 to certain specific situations in Chapters 2 and 3. Other requisite details, as and when necessary, will be set out in those chapters.

1.4. Algebraic Properties

One ready consequence of our discussion in the previous section is that cardinal interpolation by ψ is correct only if there is no non-trivial sequence which interpolates the zero-data; i.e., there is no non-trivial sequence a for which (1.3.2) holds. Therefore, if cardinal interpolation by ψ is correct, then it is necessary that the following algebraic condition hold:

$$\begin{aligned} &\sum_{j\in\mathbf{Z}^s} a(j)\psi(\cdot - j) = 0, \qquad a(j) \in \mathbf{C}, \\ &\Rightarrow \qquad a(j) = 0 \qquad \forall j \in \mathbf{Z}^s. \end{aligned} \tag{1.4.1}$$

Whenever condition (1.4.1) does hold, we say that the **integer translates of ψ are linearly independent**. In order to reformulate this notion of linear independence, we define the **kernel space** associated with ψ as follows:

$$K(\psi) := \left\{ a : \mathbf{Z}^s \to \mathbf{C} : \sum_{j\in\mathbf{Z}^s} a(j)\psi(\cdot - j) = 0 \right\}. \tag{1.4.2}$$

It is quite obvious that the integer translates of ψ are linearly independent if and only if $K(\psi)$ is the trivial space $\{0\}$. Closely connected with the space $K(\psi)$ is the associated **null set** $N(\psi)$ given by

$$N(\psi) := \left\{ \theta \in \mathbf{C}^s : \ \widehat{\psi}(\theta + 2\pi j) = 0 \text{ for all } j \in \mathbf{Z}^s \right\}. \tag{1.4.3}$$

The actual connexions between $K(\psi)$ and $N(\psi)$ and consequences thereof will be made precise in Chapter 4.

Let us now dwell at some length on the linear independence of box spline translates. The situation here is best understood only when the defining matrix Ξ is an integer matrix (i.e., all entries in Ξ are integers). The relevant theorems in this regard will be stated in Chapter 4. It is a consequence of one such theorem [BH82, Proposition 4] (also see Theorem 4.2.4 in Chapter 4) that the integer translates of the bivariate box spline on the four direction mesh are linearly dependent. However, Jetter and Riemenschneider [JR87] have shown that the translates, when restricted to the submodule of $\mathbf{Z}^2$ generated by the vectors $(1,1)$ and $(1,-1)$, are in fact linearly independent. This result motivates our study in Chapter 4.

In order to formulate the general problem considered therein, let $\mathcal{A}$ be a submodule of the $\mathbf{Z}$-module $\mathbf{Z}^s$ and B_Ξ, the polynomial box spline associated with a given $s \times n$ integer matrix Ξ. We define the space

$$K_{\mathcal{A}}(B_\Xi) := \left\{ a : \mathcal{A} \to \mathbf{C} : \ \sum_{k \in \mathcal{A}} a(k) B_\Xi(\cdot - k) = 0 \right\} \tag{1.4.4}$$

and say that the **$\mathcal{A}$-translates of B_Ξ are linearly independent** if $K_{\mathcal{A}}(B_\Xi) = \{0\}$. The question which then arises is the following: For what submodules $\mathcal{A}$ is the space $K_{\mathcal{A}}(B_\Xi)$ trivial?

If $\mathcal{A}$ is generated by a non-singular $s \times s$ integer matrix A, i.e., $\mathcal{A} = A\mathbf{Z}^s$, then it turns out (details are spelled out in Chapter 4) that $K_{\mathcal{A}}(B_\Xi)$ reduces to the trivial space precisely when the kernel space $K(B_{A^{-1}\Xi})$ (as in (1.4.2)) is trivial. In general, $A^{-1}\Xi$ is a rational matrix and may well have non-integer entries; this leads us to consider box splines associated with rational matrices. These considerations occupy the bulk of Chapter 4 where we study the linear independence of such box spline translates.

The final chapter concentrates on the finite dimensionality of the kernel space $K(B_{\Xi,\lambda})$ corresponding to a complex exponential box spline associated with a rational matrix Ξ. Results of this nature have been obtained by Dahmen and Micchelli [DM85], and Dahmen, Jia, and Micchelli [DJM88, DJM89-1, DJM89-2] for box splines associated with integer matrices. These results serve as both the motivation and starting point for our own investigations. Our intention, once again, is to provide a comparable development for rational matrices.

It is clear from the foregoing discussions that the focus of interest in the second half of this thesis is the theory of box splines associated with rational matrices. This theory, in stark contrast with that of box splines associated with integer matrices, remains largely uncharted territory. Our efforts here constitute a few short steps towards its eventual development. Although our aim has been to extend some known results involving integer matrices to the setting of

rational matrices, the methods we have used are discernibly different. Even so, our attempts are far from being complete. We have not been able to provide precise characterizations even where suitable ones do exist for integer matrices. Our theme is basically one of finding appropriate sufficient conditions for certain phenomena to hold. Usually these conditions are not necessary unless the matrix in question has only integer entries. Based on our study (and our methods) so far, obtaining such exact characterizations in the case of rational matrices appears to be rather difficult.

CHAPTER 2

UNIVARIATE CARDINAL INTERPOLATION BY SHIFTED B–SPLINES

2.1. Introduction

In this chapter we study cardinal interpolation of bounded data by integer translates of shifted univariate B-splines. In the first part of the chapter, we use Fourier analytic methods to recover a known result which characterizes the corrrectness of cardinal interpolation by such splines. Aided by this, we establish a convergence theorem for the class of shifted B-splines in the second part of the chapter.

2.2. Correctness of Cardinal interpolation

Let B_n denote the centered univariate B-spline of order n. If $\alpha \in \mathbf{R}$, then we define the function $B_{n,\alpha}$ by $B_{n,\alpha}(\cdot) := B_n(\cdot + \alpha)$. The corresponding characteristic polynomial (recall (1.3.3)) is therefore given by

$$P_{n,\alpha}(x) := \sum_{j \in \mathbf{Z}} B_{n,\alpha}(j) e^{-ijx}. \tag{2.2.1}$$

The purpose of this section is to provide an alternate proof of the following theorem due to Micchelli [M76] and de Boor and Schoenberg [BS76].

Theorem 2.2.2. *Let $n \in \mathbf{N}$ and $\alpha \in [-1/2, 1/2]$ be fixed. Then, cardinal interpolation with $B_{n,\alpha}$ is correct if and only if $\alpha \in (-1/2, 1/2)$.*

We wish to recover the above result using Theorem 1.3.4. Indeed, in view of the latter, it is easy to see that Theorem 2.2.2 is a direct consequence of the following

Theorem 2.2.3. *Let $n \in \mathbf{N}$ and $\alpha \in [-1/2, 1/2]$ be fixed. Then,*

(a) $P_{n,\alpha}(x) \neq 0$ if $-\pi \leq x \leq \pi$ and $-1/2 < \alpha < 1/2$ or if $-\pi < x < \pi$ and $\alpha = \pm 1/2$.

(b) $P_{n,1/2}$ and $P_{n,-1/2}$ have simple zeroes at $x = \pm\pi$ respectively.

The proof relies on the following lemma.

Lemma 2.2.4. *Let $0 \leq b < a \leq 1$ and $-\pi \leq \theta \leq \pi$. Then, for any positive integer j,*

$$h(\theta) := \left| \frac{1 + ae^{i(2j+1)\theta}}{1 + be^{i\theta}} \right|$$
$$= \left[\frac{1 + a^2 + 2a\cos(2j+1)\theta}{1 + b^2 + 2b\cos\theta} \right]^{\frac{1}{2}} \leq 4(2j+1).$$

Proof. Since $h(\theta)$ is even, it suffices to consider $\theta \in [0, \pi]$. There are two possible cases.

Case(i). $0 \leq \theta \leq \pi/2$. Then

$$h^2(\theta) \leq (1+a)^2 \leq 4. \tag{2.2.5}$$

Case(ii). $\pi/2 < \theta \leq \pi$. Setting $\eta = \pi - \theta$, it is easy to see that

$$h^2(\theta) = \frac{1 + a^2 - 2a\cos(2j+1)\eta}{1 + b^2 - 2b\cos\eta}.$$

Since $1 - \phi^2/2 \leq \cos\phi \leq 1 - \phi^2/2 + \phi^4/24$ for $\phi \geq 0$, it follows that

$$h^2(\theta) \leq \frac{1 + a^2 - 2a[1 - (2j+1)^2\eta^2/2]}{1 + b^2 - 2b[1 - \eta^2/2 + \eta^4/24]}$$
$$= \frac{(1-a)^2 + a(2j+1)^2\eta^2}{(1-b)^2 + b\eta^2[1 - \eta^2/12]}.$$

Observing that $0 \le \eta < \pi/2$ and $b < a \le 1$, it is clear that when $0 \le b \le 1/2$,

$$h^2(\theta) \le \frac{(1-a)^2}{(1-b)^2} + \frac{(2j+1)^2\pi^2/4}{(1-b)^2} \le 1 + \pi^2(2j+1)^2. \tag{2.2.6}$$

Similarly, if $1/2 \le b < a < 1$, then we have

$$h^2(\theta) \le \frac{(1-a)^2}{(1-b)^2} + \frac{(2j+1)^2}{b[1-\eta^2/12]} \le 1 + \pi^2(2j+1)^2. \tag{2.2.7}$$

The required inequality follows from (2.2.5), (2.2.6), and (2.2.7).

♠

Proof of Theorem 2.2.3. Firstly, we note that the evenness of B_n guarantees

$$P_{n,\alpha}(-x) = P_{n,-\alpha}(x), \qquad x \in \mathbf{R}. \tag{2.2.8}$$

So it suffices to consider $x \in [0, \pi]$ whilst proving (a).

Now, applying the Poisson summation formula to (2.2.1), we have

$$\begin{aligned} P_{n,\alpha}(x) &= \sum_{j\in\mathbf{Z}} \widehat{B}_{n,\alpha}(x + 2\pi j) \\ &= \sum_{j\in\mathbf{Z}} \widehat{B}_n(x + 2\pi j) e^{i\alpha(x+2\pi j)}. \end{aligned} \tag{2.2.9}$$

Set $x = 2\pi u$ so that $0 \le u \le 1/2$. Then (2.2.9) reads as follows:

$$e^{-i\alpha 2\pi u} P_{n,\alpha}(2\pi u) = \sum_{j\in\mathbf{Z}} \widehat{B}_n(2\pi u + 2\pi j) e^{i\alpha 2\pi j}. \tag{2.2.10}$$

Recall from (1.2.3) that

$$\widehat{B}_n(t) = \left[\frac{\sin(t/2)}{(t/2)}\right]^n, \qquad t \in \mathbf{R}.$$

Using this and the fact that $\widehat{B}_n$ is even in (2.2.10), we have

$$e^{-i\alpha 2\pi u}P_{n,\alpha}(2\pi u) = \widehat{B}_n(2\pi u)\left[1 + \left(\frac{u}{1-u}\right)^n e^{-i\alpha 2\pi} + \sum_{j=1}^{\infty}(-1)^{jn}\left(\frac{u}{u+j}\right)^n e^{i\alpha 2\pi j} + \sum_{j=2}^{\infty}(-1)^{jn}\left(\frac{u}{u-j}\right)^n e^{-i\alpha 2\pi j}\right].$$

After some simplification, the expression above reduces to

$$e^{-i\alpha 2\pi u}P_{n,\alpha}(2\pi u) = \widehat{B}_n(2\pi u)\left[1 + \left(\frac{u}{1-u}\right)^n e^{-i\alpha 2\pi}\right] \times \left[1 + \sum_{j=1}^{\infty}(-1)^{jn} e^{i\alpha 2\pi j}\left(\frac{u}{u+j}\right)^n A_{n,\alpha,j}(u)\right] \tag{2.2.11}$$

where, for brevity,

$$A_{n,\alpha,j}(u) := \frac{1 + \left(\frac{j+u}{j+1-u}\right)^n e^{-i\alpha 2\pi(2j+1)}}{1 + \left(\frac{u}{1-u}\right)^n e^{-i\alpha 2\pi}}. \tag{2.2.12}$$

Since $\widehat{B}_n(2\pi u)$ is non-zero for $0 \le u \le 1/2$ while

$$1 + \left(\frac{u}{1-u}\right)^n e^{-i\alpha 2\pi} = 0 \quad \text{iff} \quad \alpha = \pm 1/2 \text{ and } u = 1/2,$$

it suffices to prove that the remaining factor in (2.2.11) is non-zero for $0 \le u \le 1/2$ and $-1/2 \le \alpha \le 1/2$ in order to establish the desired result. In turn, this will follow readily once we show that

$$\sum_{j=1}^{\infty}\left|\frac{u}{u+j}\right|^n |A_{n,\alpha,j}(u)| < 1 \quad 0 \le u \le 1/2, \quad -1/2 \le \alpha \le 1/2. \tag{2.2.13}$$

At the outset, it is clear from Lemma 2.2.4 and (2.2.12) that $|A_{n,\alpha,j}(u)| \leq 4(2j+1)$ for $0 \leq u \leq 1/2$ and $-1/2 \leq \alpha \leq 1/2$. As

$$\left|\frac{u}{u+j}\right| \leq \frac{1}{2j+1} \quad \text{for } 0 \leq u \leq 1/2,$$

it follows immediately that

$$\begin{aligned} \sum_{j=1}^{\infty}\left|\frac{u}{u+j}\right|^{n}|A_{n,\alpha,j}(u)| &\leq 4\sum_{j=1}^{\infty}\left(\frac{1}{2j+1}\right)^{n-1} \\ &\leq C < 1 \qquad \text{for } n \geq 3. \end{aligned} \tag{2.2.14}$$

It should be noted that C is independent of u, n, and α. This proves (2.2.13) for $n \geq 3$.

The theorem can be checked directly for $n = 1, 2$ and is therefore proved in its entirety.

♠

Remark. Recently, Smith and Ward [SW90] have shown that for $-\pi \leq x \leq \pi$ and $-1/2 \leq \alpha \leq 1/2$, we have $|1 - P_{n,\alpha}(x)| \leq 1$ with equality if and only if $|x| = \pi$ and $|\alpha| = 1/2$.

2.3. A Convergence theorem

Suppose that f is a bounded function defined on $\mathbf{R}$ and let $I_{n,\alpha}f$ denote the interpolant $\sum_{j\in\mathbf{Z}} a_j B_{n,\alpha}(\cdot - j)$ which agrees with f on $\mathbf{Z}$. As is well known, $I_{n,\alpha}f$ can also be expressed in its *Lagrange form*:

$$I_{n,\alpha}f =: \sum_{j\in\mathbf{Z}} f(j)L_{n,\alpha}(\cdot - j).$$

The function $L_{n,\alpha}$ is called the *fundamental function* of interpolation and its Fourier transform is given by

$$\widehat{L}_{n,\alpha}(x) = \frac{\widehat{B}_{n,\alpha}(x)}{P_{n,\alpha}(x)}. \tag{2.3.1}$$

The main objective of this section is to obtain a "degree tends to infinity" theorem of the Schoenberg type [S73, Rie76] for the class of shifted splines.

In what follows, $K(\alpha)$ will stand for a constant dependent only on α. It should be remarked, however, that its actual numerical value may differ at each appearance.

Our quest begins with the following preliminary lemma.

Lemma 2.3.2. *Let* $-1/2 < \alpha < 1/2$ *and* $0 < \delta < \pi/2$. *Then the following hold:*

(a) $\lim_{n\to\infty} \widehat{L}_{n,\alpha}(x) = 1$ *for* $-\pi < x < \pi$ *and the convergence is uniform on compact subintervals of* $(-\pi, \pi)$.

(b) $\lim_{n\to\infty} \widehat{L}_{n,\alpha}(\pm\pi) = e^{\pm i\alpha\pi}/(2\cos\pi\alpha)$.

(c) *For* $x \in [-\pi - \delta, \pi + \delta]$, $|\widehat{L}_{n,\alpha}(x)| \le K(\alpha)$.

(d) *For* $\pi + \delta < |x| = 2\pi(u + j)$, $u \in [-1/2, 1/2]$ *and* $j = 1, 2, 3, ...,$

$$|\widehat{L}_{n,\alpha}(x)| \le \begin{cases} K(\alpha)[(\pi - \delta)/(\pi + \delta)]^n, & \text{if } j = 1; \\ K(\alpha)(2j-1)^{-n}, & \text{if } j = 1, 2, 3, ... \end{cases}$$

Proof. (a) By virtue of (2.3.1), (2.2.8), and the evenness of $\widehat{B}_n$, it follows that

$$\widehat{L}_{n,\alpha}(-x) = \widehat{L}_{n,-\alpha}(x). \tag{2.3.3}$$

So we may assume that $x \in [0, \pi)$. Now (a) follows easily from (2.3.1), (2.2.11), and (2.2.14).

Assertion (b) is an easy consequence of (2.3.1), (2.2.11), (2.2.14), and (2.3.3).

(c) To begin with, let $x \in [0, \pi]$. Then $x = 2\pi u$ for $u \in [0, 1/2]$. For such u, it is not hard to see that

$$\left|1 + \left(\frac{u}{1-u}\right)^n e^{-i\alpha 2\pi}\right| \geq \begin{cases} 1, & \text{if } -1/4 \leq \alpha \leq 1/4; \\ |\sin 2\pi\alpha|, & \text{if } \pm\alpha \in (1/4, 1/2). \end{cases} \tag{2.3.4}$$

The preceding estimate, taken in conjunction with (2.2.11) and (2.2.14), gives (c) for $x \in [0, \pi]$. The obvious symmetry of the lower bounds (w.r.t.α) in (2.3.4), coupled with (2.3.3), gives (c) for $x \in [-\pi, 0]$ as well. Now, for

$$|x| = 2\pi(u + j), \quad j = 1, 2, ...,$$

the periodicity of $P_{n,\alpha}$ permits the estimate

$$\begin{aligned} \left|\widehat{L}_{n,\alpha}(|x|)\right| &= \left|\frac{\widehat{B}_{n,\alpha}(2\pi u + 2\pi j)}{P_{n,\alpha}(2\pi u + 2\pi j)} \frac{\widehat{B}_{n,\alpha}(2\pi u)}{\widehat{B}_{n,\alpha}(2\pi u)}\right| \\ &= |\widehat{L}_{n,\alpha}(2\pi u)| \left|\frac{u}{u+j}\right|^n \\ &\leq \begin{cases} K(\alpha)[(\pi - \delta)/(\pi + \delta)]^n, & \text{if } j = 1, |x| > \pi + \delta; \\ K(\alpha)(2j-1)^{-n}, & \text{if } j \geq 1. \end{cases} \end{aligned}$$

The remaining assertions of the Lemma follow from this and (2.3.3).

♠

The convergence theorem can now be stated.

Theorem 2.3.5. *Let*

$$f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixt}\, d\mu(t)$$

for a bounded measure μ on $[-\pi,\pi)$ and let $-1/2 < \alpha < 1/2$.

(a) If μ is absolutely continuous (w.r.t. Lebesgue measure), then $I_{n,\alpha}f$ converges uniformly to f.

(b) If $\mu = \delta_{-\pi}$, then $I_{n,\alpha}f$ converges uniformly to $\cos\pi(x+\alpha)/(\cos\pi\alpha)$.

Proof. (a) Since $\{f(-j) : j \in \mathbf{Z}\}$ are the Fourier coefficients of the measure μ, it follows from the weak convergence of its Fourier series that

$$\begin{aligned} f(x) - I_{n,\alpha}f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ixt}\, d\mu(t) - \sum_{j\in\mathbf{Z}} f(j)L_{n,\alpha}(x-j) \\ &= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} e^{ixt}\, d\mu(t) - \sum_{j\in\mathbf{Z}} f(j)\int_{-\infty}^{\infty} e^{-ijt}\widehat{L}_{n,\alpha}(t)e^{ixt}\, dt\right] \quad (2.3.6) \\ &= \frac{1}{2\pi}\left[\int_{-\pi}^{\pi} e^{ixt}\, d\mu(t) - \int_{-\infty}^{\infty} \widehat{L}_{n,\alpha}(t)e^{ixt}\, d\tilde{\mu}(t)\right], \end{aligned}$$

where $\tilde{\mu}$ is the periodic extension of μ.

For a given $\epsilon > 0$, let $N_{-\pi} := [-\pi-\delta, -\pi+\delta]$ and $N_{\pi} := [\pi-\delta, \pi+\delta]$ be chosen such that

$$\tilde{\mu}(N_{\pm\pi}) < \epsilon \quad (2.3.7)$$

by the absolute continuity of μ.

From (2.3.6), it is clear that

$$\begin{aligned} |f(x) - I_{n,\alpha}f(x)| \le \frac{1}{2\pi}\Bigg[&\int_{-\pi+\delta}^{\pi-\delta} |1 - \widehat{L}_{n,\alpha}(t)|\, d\mu + \int_{N_{-\pi}\cup N_{\pi}} \left[1 + |\widehat{L}_{n,\alpha}(t)|\right] d\tilde{\mu} \\ &+ \int_{-\infty}^{-\pi-\delta} |\widehat{L}_{n,\alpha}(t)|\, d\tilde{\mu} + \int_{\pi+\delta}^{\infty} |\widehat{L}_{n,\alpha}(t)|\, d\tilde{\mu} + \Bigg]. \end{aligned} \quad (2.3.8)$$

Noticing that the sum of the last two integrals on the right hand side of (2.3.8) can be written as

$$\int_{-3\pi}^{-\pi-\delta} |\widehat{L}_{n,\alpha}(t)|\, d\tilde{\mu} + \int_{\pi+\delta}^{3\pi} |\widehat{L}_{n,\alpha}(t)|\, d\tilde{\mu} + \sum_{j=2}^{\infty}\left[\int_{(2j-1)\pi}^{(2j+1)\pi} |\widehat{L}_{n,\alpha}(t)|\, d\tilde{\mu} + \int_{-(2j+1)\pi}^{-(2j-1)\pi} |\widehat{L}_{n,\alpha}(t)|\, d\tilde{\mu}\right],$$

and then using (a), (c), and (d) of Lemma 2.3.2 along with (2.3.7), it follows that

$$\limsup_{n\to\infty} \|f_\alpha - I_{n,\alpha} f\|_\infty \le \frac{1}{\pi}[1 + K(\alpha)]\epsilon,$$

from which the desired conclusion follows.

(b) When $\mu = \delta_{-\pi}$,

$$I_{n,\alpha} f(x) = \widehat{L}_{n,\alpha}(-\pi)e^{-i\pi x} + \widehat{L}_{n,\alpha}(\pi)e^{i\pi x} + \sum_{j\in\mathbf{Z}\setminus\{0,1\}} \widehat{L}_{n,\alpha}((2j-1)\pi)e^{i(2j-1)\pi x}$$

which converges uniformly to

$$\frac{e^{-i\pi(x+\alpha)} + e^{i\pi(x+\alpha)}}{2\cos\pi\alpha} = \frac{\cos\pi(x+\alpha)}{\cos\pi\alpha}$$

by (b) and (d) of Lemma 2.3.2.

CHAPTER 3

BIVARIATE CARDINAL INTERPOLATION BY SHIFTED BOX SPLINES

3.1. Introduction

Our object of study in this chapter is the bivariate box spline corresponding to the directions $(1,0)$, $(0,1)$, and $(1,1)$. (Recall Example 1.2.5.) In particular, we are interested in the spline $B_{n,n,n}$ i.e., when each direction occurs with equal mulitiplicity $n \in \mathbf{N}$. Hereafter, for notational ease, we shall denote $B_{n,n,n}$ simply by B_n. Although this is in obvious conflict with the notation of the previous chapter, no confusion should arise because the univariate B-spline is *not* mentioned throughout this chapter. Likewise, if $\alpha := (\alpha_1, \alpha_2) \in \mathbf{R}^2$, then we denote the function $B_{n,n,n}(\cdot + \alpha)$ by $B_{n,\alpha}$ and the associated characteristic polynomial $P_{n,n,n,\alpha}$ by

$$P_{n,\alpha}(x) := \sum_{j \in \mathbf{Z}^2} B_{n,\alpha}(j) e^{-ijx}, \qquad x \in \mathbf{R}^2.$$

Once again, the Poisson summation formula yields

$$\begin{aligned} P_{n,\alpha}(x) &= \sum_{j \in \mathbf{Z}^2} \widehat{B}_{n,\alpha}(x + 2\pi j) \\ &= \sum_{j \in \mathbf{Z}^2} \widehat{B}_n(x + 2\pi j) e^{i\alpha(x+2\pi j)}. \end{aligned} \tag{3.1.1}$$

If $\alpha = 0$, then the correctness of cardinal interpolation by box splines corresponding to the three-direction mesh was first established by de Boor, Höllig, and Riemenschneider in [BHR85]. A special case of the result there is the

correctness of cardinal interpolation by B_n. Our interest here is in a "shifted" version of this result. In the subsequent sections, we intend to determine a shift region for α such that cardinal interpolation with $B_{n,\alpha}$ is correct whenever α belongs to this region and n is even.

3.2. Symmetries of $B_{n,\alpha}$ and $P_{n,\alpha}$

We first deduce some symmetries of $B_{n,\alpha}$ and $P_{n,\alpha}$ which serve to make further analysis less onerous. In doing so, we essentially follow [BHR85] and the (omitted) proofs of our assertions are very similar to the ones there.

Let $\mathbf{A}$ denote the group of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$$

that is isomorphic to $\mathbf{S}_3$ (the group of permutations on 3 indices). Then, for any $A \in \mathbf{A}$ and $x \in \mathbf{R}^2$, the following hold:

$$B_{n,\pm A\alpha}(\pm Ax) = B_{n,\alpha}(x) \tag{3.2.1}$$

$$\widehat{B}_{n,\pm A\alpha}(x) = \widehat{B}_{n,\alpha}(\pm A^* x) \tag{3.2.2}$$

$$P_{n,\pm A\alpha}(x) = P_{n,\alpha}(\pm A^* x) \tag{3.2.3}$$

where A^* denotes the transpose of A.

3.3. Correctness of Cardinal interpolation

We begin this central section by introducing some more notation. For convenience, we set $x := (2\pi u, 2\pi v)$ and $\theta := (\theta_1, \theta_2) := (2\pi\alpha_1, 2\pi\alpha_2)$. Let Ω and Λ denote the interiors of the regions in Figures 3.3.I and 3.3.II respectively. Let $\widetilde{\Omega}$ represent the interior of the quadrilateral whose vertices are

$$(0,0),\ (1/2,0),\ (1/3,1/3),\ \text{and}\ (0,1/2)$$

(see Figure 3.3.III) and let the (closed) triangle (see Figure 3.3.III) with vertices at

$$(0,0),\ (1/2,0),\ \text{and}\ (1/3,1/3)$$

be denoted by Δ. We further subdivide Δ into two parts, Δ_1 and Δ_2 (Figure xx), where Δ_1 is the quadrilateral formed by the vertices

$$(0,0),\ (19/48,0),\ (1/3,1/6),\ \text{and}\ (5/18,5/18)$$

and Δ_2 is the complement of Δ_1 in Δ. These regions Δ_1 and Δ_2 are chosen so that the estimates below (in Theorem 3.3.2) hold even for $n = 2$. Likewise (see Figure 3.3.II), Λ is divided into six sub-regions.

It is known [BHR85, BHR87] that $\overline{\Omega}$ (the closure of Ω) is a **fundamental domain**, i.e., its translates form an essentially disjoint partition of $\mathbf{R}^2/2\pi$. Further, Ω and $\overline{\Omega}$ are *invariant under* $\mathbf{A}^*$ as are Λ and $\bar{\Lambda}$ *under* $\mathbf{A}$.

Using (1.2.7) in (3.1.1), we see that

$$\begin{aligned} e^{-i(u\theta_1+v\theta_2)}P_{n,\alpha}(2\pi u, 2\pi v) &= \Big[S(2\pi u)S(2\pi v)S(2\pi(u+v))\Big]^n \times \\ &\times \left[1 + \sum_{(k,l)\in\mathbf{Z}^2\setminus(0,0)} \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n e^{i(k\theta_1+l\theta_2)}\right] \\ &=: \Big[S(2\pi u)S(2\pi v)S(2\pi(u+v))\Big]^n \Big[Q_{n,\theta}(u,v)\Big], \end{aligned} \tag{3.3.1}$$

where S is the sinc function defined in (1.2.4).

This section is arranged as follows. We first show that $P_{n,\alpha}(2\pi u, 2\pi v)$ does not vanish on $\overline{\Omega}$ (and hence in $\mathbf{R^2}$) for $\theta \in \Lambda$ and n even. Then we prove that for all n, $P_{n,\alpha}(2\pi u, 2\pi v)$ vanishes somewhere on the boundary of Ω for all values of θ on the boundary of Λ. Finally, using these results along with Theorem 1.3.4, we obtain a correctness result for $B_{n,\alpha}$ when n is even.

We make some preliminary observations. It is not difficult to see that $\overline{\Omega} = \bigcup_{A \in \mathbf{A}} A^* \overline{\overline{\Omega}}$. Taken together with (3.2.3) and the invariance of Λ under $\mathbf{A}$, this allows us to only consider $(u, v) \in \overline{\overline{\Omega}}$ and $\theta \in \Lambda$ in our subsequent discussion. Secondly, from the very definition of $S(t)$, it is quite clear that $\left[S(2\pi u)S(2\pi v)S(2\pi(u+v))\right]^n$ is never zero for $(u, v) \in \overline{\overline{\Omega}}$. Therefore, the zeroes of $P_{n,\alpha}$ are completely determined by $Q_{n,\theta}(u, v)$.

We are now ready for our main theorem.

Theorem 3.3.2. *Let $n \in \mathbf{N}$ be even and $\theta \in \Lambda$. Then, for $(u, v) \in \overline{\overline{\Omega}}$, $Q_{n,\theta}(u, v)$ is non-zero.*

Proof. To simplify matters, we first observe that (see (3.3.1))

$$Q_{n,(\theta_2,\theta_1)}(v, u) = Q_{n,(\theta_1,\theta_2)}(u, v). \tag{3.3.3}$$

This observation, coupled with the fact that $\overline{\overline{\Omega}}$ and Λ are symmetric about their main diagonals ($u = v$ and $\theta_1 = \theta_2$ respectively), permits us to restrict ourselves to $(u, v) \in \Delta$ and $\theta \in \Lambda$ for the duration of the proof. Let us denote the imaginary part of $Q_{n,\theta}$ by

$$I_{n,\theta}(u, v) := \mathrm{Im}[Q_{n,\theta}(u, v)] \tag{3.3.4}$$

and the imaginary part of $e^{i\theta_2}Q_{n,\theta}(u,v)$ by

$$\tilde{I}_{n,\theta}(u,v) := \mathrm{Im}[e^{i\theta_2}Q_{n,\theta}(u,v)]. \tag{3.3.5}$$

The proof proceeds in a series of four steps. The actual numerical bounds for the various infinite series that follow were generated by a computer and are accurate to at least the number of decimal places shown.

Step I.

$$(u,v) \in \Delta, \quad \theta = (0,0).$$

In this case, $Q_{n,\theta}(u,v)$ is ***strictly positive*** by [BHR85, Theorem 4].

Step II.

$$(u,v) \in \Delta_1, \quad \theta \in \Lambda \setminus (0,0).$$

From Figure 3.3.III, we see that $(u,v) \in \Delta_1$ implies

$$0 \le u \le 19/48; \ 0 \le v \le 5/18; \ \text{and } 0 \le u+v \le 5/9. \tag{3.3.6}$$

We first note that for $0 \le a \le w \le b < 1$, $j \in \mathbf{Z}$, and $n(\ge 2) \in \mathbf{N}$, one has

$$\left|\frac{w}{w+j}\right|^n \le \left|\frac{b}{b+j}\right|^n; \tag{3.3.7}$$

a consequence of the fact that the function $g(w) := \left|\frac{w}{j+w}\right|$ is non-decreasing on $[a,b]$. Now, observe that

$$\begin{aligned}
|Q_{n,\theta}(u,v)| &\ge 1 - \sum_{(k,l)\in\mathbf{Z}^2\setminus(0,0)} \left|\frac{u}{u+k}\right|^n \left|\frac{v}{v+l}\right|^n \left|\frac{u+v}{u+v+k+l}\right|^n \\
&\ge 1 - \sum_{(k,l)\in\mathbf{Z}^2\setminus(0,0)} \left|\frac{19}{48k+19}\right|^n \left|\frac{5}{18l+5}\right|^n \left|\frac{5}{9(k+l)+5}\right|^n \\
&\ge 1 - \sum_{(k,l)\in\mathbf{Z}^2\setminus(0,0)} \left|\frac{19}{48k+19}\right|^2 \left|\frac{5}{18l+5}\right|^2 \left|\frac{5}{9(k+l)+5}\right|^2 \\
&\ge 1 - 0.9939\ldots > 0,
\end{aligned} \tag{3.3.8}$$

by (3.3.6), (3.3.7), and the fact that the summands on the right are less than 1.

So, Step II is complete.

The next two steps (III and IV) are less straightforward, as we shall presently see.

Step III.

$$(u,v) \in \Delta_2, \quad \theta \in [\Lambda_1 \cup \Lambda_2] \setminus (0,0).$$

We do the case $\theta \in \Lambda_1$ explicitly. The case $\theta \in \Lambda_2$ can be handled entirely analogously. Firstly, from Figure 3.3.III, we have for all $(u,v) \in \Delta_2$,

$$5/18 \le u \le 1/2;\ 0 \le v \le 1/3;\ 19/48 \le u+v \le 2/3;\ v \le u. \tag{3.3.9}$$

Also, from Figure 3.3.II, one can see that $\theta \in \Lambda_1$ implies

$$0 \le \theta_2 \le \theta_1 \le \pi. \tag{3.3.10}$$

In addition to (3.3.7), we will also use the following inequality:

$$\text{For } 0 \le a \le w \le b < 1,\ j \in \mathbf{Z} \setminus \{0\},\ n(\ge 2) \in \mathbf{N},$$

$$\left|\frac{1-w}{j+w}\right|^n \le \left|\frac{1-a}{j+a}\right|^n. \tag{3.3.11}$$

This follows from the observation that the function $h(w) := \left|\frac{1-w}{j+w}\right|$ is non-increasing in the interval $[a,b]$. Furthermore, we also have, for $m \in \mathbf{Z}$ and $\phi \in \mathbf{R}$,

$$\begin{aligned} |\cos\phi| &\le 1 \\ |\sin m\phi| &\le |m \sin\phi|. \end{aligned} \tag{3.3.12}$$

We now show that $Q_{n,\theta}(u,v)$ has no zeroes by showing that $I_{n,\theta}(u,v)$ has none. By virtue of (3.3.1), we have

$$\begin{aligned} I_{n,\theta}(u,v) &= \sum_{(k,l)\in\mathbf{Z}^2} \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \sin(k\theta_1 + l\theta_2) \\ &= \sum_{k+l\neq 0} \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \sin k\theta_1 \cos l\theta_2 \\ &+ \sum_{k+l\neq 0} \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \cos k\theta_1 \sin l\theta_2 \\ &+ \sum_{k+l=0} \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \sin(k\theta_1 + l\theta_2) \\ &=: S' + S'' + S''' \ . \end{aligned} \tag{3.3.13}$$

We shall handle each of these three sums separately by isolating an appropriate dominant term from each of them and factoring it out. If $\tau(k,l)$ represents (k,l)-th term in (3.3.13), then the dominant terms for S', S'', and S''' are $\tau(-1,0)$, $\tau(0,-1)$ and $\tau(1,-1)+\tau(-1,1)$ respectively. Thus, we have

$$\begin{aligned} S' &= -\left(\frac{u}{1-u}\right)^n \left(\frac{u+v}{1-u-v}\right)^n \sin\theta_1 \times \\ &\times \left[1 - \sum{}' \left(\frac{1-u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{1-u-v}{u+v+k+l}\right)^n \frac{\sin k\theta_1}{\sin\theta_1} \cos l\theta_2\right], \end{aligned} \tag{3.3.14}$$

$$\begin{aligned} S'' &= -\left(\frac{v}{1-v}\right)^n \left(\frac{u+v}{1-u-v}\right)^n \sin\theta_2 \times \\ &\times \left[1 - \sum{}'' \left(\frac{u}{u+k}\right)^n \left(\frac{1-v}{v+l}\right)^n \left(\frac{1-u-v}{u+v+k+l}\right)^n \frac{\sin l\theta_2}{\sin\theta_2} \cos k\theta_1\right] \end{aligned} \tag{3.3.15}$$

and

$$S''' = -(-1)^n \left(\frac{u}{1-u}\right)^n \left(\frac{v}{1+v}\right)^n \sin(\theta_1-\theta_2)\left[1-\left(\frac{1-u}{1-v}\right)^n\left(\frac{1+v}{1+u}\right)^n\right]\times$$
$$\times\left[1+\sum_{r=2}^{\infty}\left(\frac{1-u}{r-u}\right)^n\left(\frac{1+v}{r+v}\right)^n a_r(u,v,\theta_1,\theta_2)\right] \tag{3.3.16}$$

where $\sum'$ denotes the sum over

$$(k,l)\in \mathbf{Z}^2 \text{ with } k+l\neq 0,\ (k,l)\neq(-1,0) \text{ and } k\neq 0,$$

$\sum''$ represents the sum taken over

$$(k,l)\in \mathbf{Z}^2 \text{ with } k+l\neq 0,\ (k,l)\neq(0,-1) \text{ and } l\neq 0$$

and

$$a_r(u,v,\theta_1,\theta_2) := \left[\frac{1-\left(\frac{r-u}{r-v}\right)^n\left(\frac{r+v}{r+u}\right)^n}{1-\left(\frac{1-u}{1-v}\right)^n\left(\frac{1+v}{1+u}\right)^n}\right]\frac{\sin r(\theta_1-\theta_2)}{\sin(\theta_1-\theta_2)}. \tag{3.3.17}$$

Let us consider the sums that occur in (3.3.14), (3.3.15) and (3.3.16) separately. Using the inequalities (3.3.7), (3.3.9), (3.3.11), and (3.3.12) repeatedly, we find that for (3.3.14),

$$\begin{aligned}
&1-\sum{}'\left(\frac{1-u}{u+k}\right)^n\left(\frac{v}{v+l}\right)^n\left(\frac{1-u-v}{u+v+k+l}\right)^n\frac{\sin k\theta_1}{\sin\theta_1}\cos l\theta_2\\
&\geq 1-\sum{}'\left|\frac{1-u}{u+k}\right|^n\left|\frac{v}{v+l}\right|^n\left|\frac{1-u-v}{u+v+k+l}\right|^n\left|\frac{\sin k\theta_1}{\sin\theta_1}\right||\cos l\theta_2|\\
&\geq 1-\sum{}'\left|\frac{13}{18k+5}\right|^n\left|\frac{1}{3l+1}\right|^n\left|\frac{29}{48(k+l)+19}\right|^n|k|\\
&\geq 1-\sum{}'\left|\frac{13}{18k+5}\right|^2\left|\frac{1}{3l+1}\right|^2\left|\frac{29}{48(k+l)+19}\right|^2|k|\\
&\geq 1-0.3342\ldots>0
\end{aligned} \tag{3.3.18}$$

and similarly, for (3.3.15),

$$
\begin{aligned}
&1-\sum{}''\left(\frac{u}{u+k}\right)^n\left(\frac{1-v}{v+l}\right)^n\left(\frac{1-u-v}{u+v+k+l}\right)^n\frac{\sin l\theta_2}{\sin\theta_2}\cos k\theta_1\\
&\geq 1-\sum{}''\left|\frac{u}{u+k}\right|^n\left|\frac{1-v}{v+l}\right|^n\left|\frac{1-u-v}{u+v+k+l}\right|^n\left|\frac{\sin l\theta_2}{\sin\theta_2}\right||\cos k\theta_2|\\
&\geq 1-\sum{}''\left|\frac{1}{2k+1}\right|^n\left|\frac{1}{l}\right|^n\left|\frac{29}{48(k+l)+19}\right|^n|l|\\
&\geq 1-\sum{}''\left|\frac{1}{2k+1}\right|^2\left|\frac{1}{l}\right|^2\left|\frac{29}{48(k+l)+19}\right|^2|l|\\
&\geq 1-0.9821...>0. \qquad (3.3.19)
\end{aligned}
$$

To estimate the sum in (3.3.16) from below, we also need the following: for $r\in\mathbf{N}$,

$$
0\leq v\leq u\Rightarrow 0\leq\left(\frac{1-u}{1-v}\right)\left(\frac{1+v}{1+u}\right)\leq\left(\frac{r-u}{r-v}\right)\left(\frac{r+v}{r+u}\right)\leq 1. \qquad (3.3.20)
$$

Using inequalities (3.3.12) and (3.3.20) in (3.3.17), we have

$$
|a_r(u,v,\theta_1,\theta_2)|\leq 1\cdot r,\qquad r\in\mathbf{N}. \qquad (3.3.21)
$$

Consequently, one has

$$
\begin{aligned}
&1+\sum_{r=2}^{\infty}\left(\frac{1-u}{r-u}\right)^n\left(\frac{1+v}{r+v}\right)^n a_r(u,v,\theta_1,\theta_2)\\
&\geq 1-\sum_{r=2}^{\infty}\left|\frac{1-u}{r-u}\right|^n\left|\frac{1+v}{r+v}\right|^n|a_r(u,v,\theta_1,\theta_2)|\\
&\geq 1-\sum_{r=2}^{\infty}\left(\frac{1}{r}\right)^n\left(\frac{4}{3r+1}\right)^n r \qquad (3.3.22)\\
&\geq 1-\sum_{r=2}^{\infty}\left(\frac{1}{r}\right)^2\left(\frac{4}{3r+1}\right)^2 r\\
&\geq 1-0.28...>0.
\end{aligned}
$$

From (3.3.18), (3.3.19), and (3.3.22), we conclude that the signs of S', S'' and S''' *coincide with those of their respective dominant terms* and that *their zeroes are precisely those of these dominant terms.* Now, from (3.3.10), (3.3.20), and the fact that n is even, one can see that *all of these three dominant terms are of the same sign* and that *they do not vanish simultaneously.* This shows that $I_{n,\theta}(u,v)$ is non-zero and finishes Step III.

Step IV.

In this final step, we exhaust the only remaining case; namely,

$$(u,v) \in \Delta_2, \quad \theta \in [\bigcup_{i=3}^{6} \Lambda_i] \setminus (0,0).$$

Note that the estimates for u, v and $u+v$ are still given by (3.3.9) as in the previous step.

We shall prove that $Q_{n,\theta}(u,v)$ has no zeroes by showing that $e^{i\theta_2}Q_{n,\theta}(u,v)$ does not have any; the latter, in turn, will be achieved by demonstrating the absence of zeroes for $\tilde{I}_{n,\theta}(u,v)$. It will become quite apparent that the *modus operandi* for doing so is similar to that in Step III.

The first part of the proof involves some tedious, albeit unavoidable calculations. We commence by writing

$$\begin{aligned}
&\tilde{I}_{n,\theta}(u,v) \\
&= \sin\theta_2 + \sum_{(k,l)\in\mathbf{Z}^2\setminus(0,0)} \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \sin(k\theta_1 + (l+1)\theta_2) \\
&\quad = \sin\theta_2 + \left(\frac{u}{1-u}\right)^n \left(\frac{u+v}{1-u-v}\right)^n \sin(\theta_2-\theta_1) + \\
&\quad + \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \cos k\theta_1 \sin(l+1)\theta_2 \\
&\quad + \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \sin k\theta_1 \cos(l+1)\theta_2
\end{aligned} \tag{3.3.23}$$

where $\sum'''$ stands for the sum over

$$(k,l) \in \mathbf{Z}^2;\ (k,l) \neq (0,0),\ (k,l) \neq (-1,0).$$

The last sum in (3.3.23) can be recast by writing $\theta_1 = \theta_2 - (\theta_2 - \theta_1)$; the resulting expression for $\tilde{I}_{n,\theta}$ is the following:

$$\begin{aligned}
&\tilde{I}_{n,\theta}(u,v) = \\
&\quad \sin\theta_2 + \left(\frac{u}{1-u}\right)^n \left(\frac{u+v}{1-u-v}\right)^n \sin(\theta_2 - \theta_1) + \\
&+ \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \cos k\theta_1 \sin(l+1)\theta_2 \\
&+ \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \sin k\theta_2 \cos k(\theta_2 - \theta_1) \cos(l+1)\theta_2 \\
&- \sum{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \sin k(\theta_2 - \theta_1) \cos k\theta_2 \cos(l+1)\theta_2.
\end{aligned} \tag{3.3.24}$$

The first two summands in (3.3.24) constitute the dominant terms in the expression for $\tilde{I}_{n,\theta}(u,v)$. Accordingly, we assign the appropriate sums in (3.3.24) to each of these (dominant) terms and then factor them out. More precisely, we write (3.3.24) as follows:

$$\begin{aligned}
&\tilde{I}_{n,\theta}(u,v) = \\
&\quad \sin\theta_2 \Bigg[1 + \\
&+ \sum_{l \neq -1}{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \frac{\sin(l+1)\theta_2}{\sin\theta_2} \cos k\theta_1 \\
&+ \sum_{k \neq 0}{}''' \left(\frac{u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{u+v}{u+v+k+l}\right)^n \frac{\sin k\theta_2}{\sin\theta_2} \cos k(\theta_2 - \theta_1) \cos(l+1)\theta_2 \Bigg]
\end{aligned}$$

$$+\left(\frac{u}{1-u}\right)^n\left(\frac{u+v}{1-u-v}\right)^n \sin(\theta_2-\theta_1)\times$$

$$\times\Bigg[1-$$

$$-\sum{}'\left(\frac{1-u}{k+u}\right)^n\left(\frac{v}{l+v}\right)^n\left(\frac{1-u-v}{k+l+u+v}\right)^n\frac{\sin k(\theta_2-\theta_1)}{\sin(\theta_2-\theta_1)}\cos k\theta_2\cos(l+1)\theta_2$$

$$-\left(\frac{v}{u+v}\right)^n\sum_{\substack{k+l=0\\k\neq 0}}\left(\frac{1-u}{k+u}\right)^n\left(\frac{1-u-v}{l+v}\right)^n\frac{\sin k(\theta_2-\theta_1)}{\sin(\theta_2-\theta_1)}\cos k\theta_2\cos(l+1)\theta_2\Bigg]$$

$$=: T' + T'' \tag{3.3.25}$$

where $\sum'$ and $\sum'''$ are the same as before.

Just as in the previous step, we estimate, from below, the sums that are featured in T' and T'' individually and show that they are bounded away from zero. Again making use of the inequalities given by (3.3.7), (3.3.9), (3.3.11) and (3.3.12), we have for T',

$$1+\sum_{l\neq -1}'''\left(\frac{u}{u+k}\right)^n\left(\frac{v}{v+l}\right)^n\left(\frac{u+v}{u+v+k+l}\right)^n\frac{\sin(l+1)\theta_2}{\sin\theta_2}\cos k\theta_1$$

$$+\sum_{k\neq 0}'''\left(\frac{u}{u+k}\right)^n\left(\frac{v}{v+l}\right)^n\left(\frac{u+v}{u+v+k+l}\right)^n\frac{\sin k\theta_2}{\sin\theta_2}\cos k(\theta_2-\theta_1)\cos(l+1)\theta_2$$

$$\geq 1-\sum_{l\neq -1}'''\left|\frac{u}{u+k}\right|^n\left|\frac{v}{v+l}\right|^n\left|\frac{u+v}{u+v+k+l}\right|^n\left|\frac{\sin(l+1)\theta_2}{\sin\theta_2}\right||\cos k\theta_1|$$

$$-\sum_{k\neq 0}'''\left|\frac{u}{u+k}\right|^n\left|\frac{v}{v+l}\right|^n\left|\frac{u+v}{u+v+k+l}\right|^n\left|\frac{\sin k\theta_2}{\sin\theta_2}\right||\cos k(\theta_2-\theta_1)\cos(l+1)\theta_2|$$

$$\geq 1-\sum_{l\neq -1}'''\left|\frac{1}{2k+1}\right|^n\left|\frac{1}{3l+1}\right|^n\left|\frac{2}{3(k+l)+2}\right|^n|l+1|$$

$$-\sum_{k\neq 0}'''\left|\frac{1}{2k+1}\right|^n\left|\frac{1}{3l+1}\right|^n\left|\frac{2}{3(k+l)+2}\right|^n|k|$$

$$\geq 1-\sum_{l\neq -1}'''\left|\frac{1}{2k+1}\right|^2\left|\frac{1}{3l+1}\right|^2\left|\frac{2}{3(k+l)+2}\right|^2|l+1|$$

$$-\sum_{k\neq 0}'''\left|\frac{1}{2k+1}\right|^2\left|\frac{1}{3l+1}\right|^2\left|\frac{2}{3(k+l)+2}\right|^2|k|$$

$$\geq 1-0.9812\ldots>0. \tag{3.3.26}$$

Turning to T'', we use the additional fact that for $n(\geq 2) \in \mathbf{N}$,

$$0 \leq v \leq u \Rightarrow 0 \leq \left(\frac{v}{u+v}\right)^n \leq \left(\frac{1}{2}\right)^n \leq \frac{1}{4}$$

to deduce that

$$\begin{aligned}
&1 - \sum{}' \left(\frac{1-u}{u+k}\right)^n \left(\frac{v}{v+l}\right)^n \left(\frac{1-u-v}{u+v+k+l}\right)^n \frac{\sin k(\theta_2-\theta_1)}{\sin(\theta_2-\theta_1)} \cos k\theta_2 \cos(l+1)\theta_2 \\
&\quad - \left(\frac{v}{u+v}\right)^n \sum_{\substack{k+l=0 \\ k\neq 0}} \left(\frac{1-u}{u+k}\right)^n \left(\frac{1-u-v}{l+v}\right)^n \frac{\sin k(\theta_2-\theta_1)}{\sin(\theta_2-\theta_1)} \cos k\theta_2 \cos(l+1)\theta_2 \\
&\geq 1 - \sum{}' \left|\frac{1-u}{u+k}\right|^n \left|\frac{v}{v+l}\right|^n \left|\frac{1-u-v}{u+v+k+l}\right|^n \left|\frac{\sin k(\theta_2-\theta_1)}{\sin(\theta_2-\theta_1)}\right| |\cos k\theta_2 \cos(l+1)\theta_2| \\
&\quad - \left|\frac{v}{u+v}\right|^n \sum_{\substack{k+l=0 \\ k\neq 0}} \left|\frac{1-u}{u+k}\right|^n \left|\frac{1-v}{v+l}\right|^n \left|\frac{\sin k(\theta_2-\theta_1)}{\sin(\theta_2-\theta_1)}\right| |\cos k\theta_2 \cos(l+1)\theta_2| \\
&\geq 1 - \sum{}' \left|\frac{13}{18k+5}\right|^n \left|\frac{1}{3l+1}\right|^n \left|\frac{29}{48(k+l)+19}\right|^n |k| \\
&\quad - \left(\frac{1}{2}\right)^n \sum_{\substack{k+l=0 \\ k\neq 0}} \left|\frac{13}{18k+5}\right|^n \left|\frac{1}{l}\right|^n |k| \\
&\geq 1 - \sum{}' \left|\frac{13}{18k+5}\right|^2 \left|\frac{1}{3l+1}\right|^2 \left|\frac{29}{48(k+l)+19}\right|^2 |k| \\
&\quad - \frac{1}{4} \sum_{\substack{k+l=0 \\ k\neq 0}} \left|\frac{13}{18k+5}\right|^2 \left|\frac{1}{l}\right|^2 |k| \\
&\geq 1 - 0.4854\ldots > 0. \qquad (3.3.27)
\end{aligned}$$

From (3.3.26) and (3.3.27), it is clear that T' and T'' *have the same signs as their respective dominant terms* and that *their only zeroes are those of these dominant terms.* Now, for $(u,v) \in \Delta_2$ and $\theta \in \left[\bigcup_{i=3}^{6} \Lambda_i\right] \setminus (0,0)$, these dominant terms are *both of the same sign* and are *never zero simultaneously.* As a result, it follows readily that $\tilde{I}_{n,\theta}(u,v)$ is non-zero as desired. This concludes Step IV

and with it the proof of the theorem in its entirety.

♠

Remark. Since the case $n = 1$ can be handled directly, the various estimates in the proof of Theorem 3.3.2 which used the fact that $n \geq 2$ do not, in fact, rely on the evenness of n. The only stage in which the evenness is used is in Step III.

Our next result, which holds true for all n, serves to locate the zeroes of $P_{n,\alpha}(x)$.

Theorem 3.3.28. *Let $n \in \mathbf{N}$ and $0 \leq \gamma \leq 1/2$. Then the following hold:*

(i) $P_{n,(1/2,\gamma)}(\pi, 0) = 0 = P_{n,(-1/2,-\gamma)}(\pi, 0)$.

(ii) $P_{n,(\gamma,1/2)}(0, \pi) = 0 = P_{n,(-\gamma,-1/2)}(0, \pi)$.

(iii) (a) $P_{n,(\gamma,1/2)}(0, -\pi) = 0 = P_{n,(-\gamma,-1/2)}(0, -\pi)$.

(b) $P_{n,(-1/2+\gamma,\gamma)}(-\pi, \pi) = 0 = P_{n,(1/2-\gamma,-\gamma)}(-\pi, \pi)$.

(iv) (a) $P_{n,(1/2,\gamma)}(-\pi, 0) = 0 = P_{n,(-1/2,-\gamma)}(-\pi, 0)$.

(b) $P_{n,(\gamma,\gamma-1/2)}(\pi, -\pi) = 0 = P_{n,(\gamma,-\gamma+1/2)}(\pi, -\pi)$.

Proof. It suffices to prove (i) and (ii). Assertions (iii) and (iv) follow from (i) and (ii) respectively by the symmetry conditions stated in (3.2.3).

To prove (i), we set $v = 0$ in (3.3.1). The resulting expression depends only on θ_1:

$$e^{-iu\theta_1} P_{n,\alpha}(2\pi u, 0) = [S(2\pi u)]^{2n} \left[1 + \sum_{k \in \mathbf{Z} \setminus \{0\}} \left(\frac{u}{u+k} \right)^{2n} e^{ik\theta_1} \right]. \qquad (3.3.29)$$

Since the RHS of (3.3.29) is zero if and only if (see Theorem 2.2.3)

$$u = 1/2 \text{ and } \theta_1 = \pm\pi,$$

(i) is proved.

Similarly, setting $u = 0$ in (3.3.1) yields (ii).

♠

Our efforts culminate in the following

Theorem 3.3.30. *Let $n \in \mathbf{N}$ be even and $2\pi\alpha \in \bar{\Lambda}$. Then cardinal interpolation with $B_{n,\alpha}$ is correct if and only if $2\pi\alpha \in \Lambda$.*

Proof. As we remarked earlier, it is manifest from (3.3.1) and the definition of $S(t)$ that $P_{n,\alpha}(2\pi u, 2\pi v)$ is zero if and only if $Q_{n,\theta}(u,v)$ is zero. The desired result now follows from Theorems 3.3.2 and 3.3.28 via Theorem 1.3.4.

♠

3.4. Further Remarks

A scrutiny of the proof of Theorem 3.3.2 reveals that a crucial point in the argument in Step III (the only place where the evenness of n is used) is the fact that *for $\theta \in \Lambda_1$ (resp. Λ_2), $I_{n,\theta}(u,v)$ does not change sign in Δ_2*. However, one can show that this is no longer true when n is odd. Moreover, in this case the situation cannot be remedied by rotating $Q_{n,\theta}(u,v)$ by a suitable angle (as done in Step IV). As a result, our method of proof fails for splines with odd multiplicities. However, subsequent to the work reported here, the validity

of Theorem 3.3.2 (and hence that of Theorem 3.3.30) has been established by Chui, Stöckler, and Ward for odd n as well. (See [CSW89-1, CSW89-2].) It must also be mentioned that Theorem 3.3.30 was obtained independently by Stöckler [St89] for the functions

$$B_{(k,l,m),\alpha}; \quad 1 \leq k,\ l,\ m \leq 2; \ k+l+m \leq 5.$$

We close the chapter with one last remark. Suppose that α is chosen along any of the three mesh directions, i.e., α is of the form $(\gamma,0)$, $(0,\gamma)$, or (γ,γ) where $-1/2 \leq \gamma \leq 1/2$. Then, from Theorem 3.3.28 and the proof of Theorem 3.3.2 (specifically, Steps I, II and IV) and the fact that the set $\Big\{(\gamma,0),\ (0,\gamma),\ (\gamma,\gamma)\Big\}$ is invariant under $\mathbf{A}$, it follows that for any n, cardinal interpolation with $B_{n,\alpha}$ is correct if and only if $-\pi < 2\pi\gamma < \pi$.

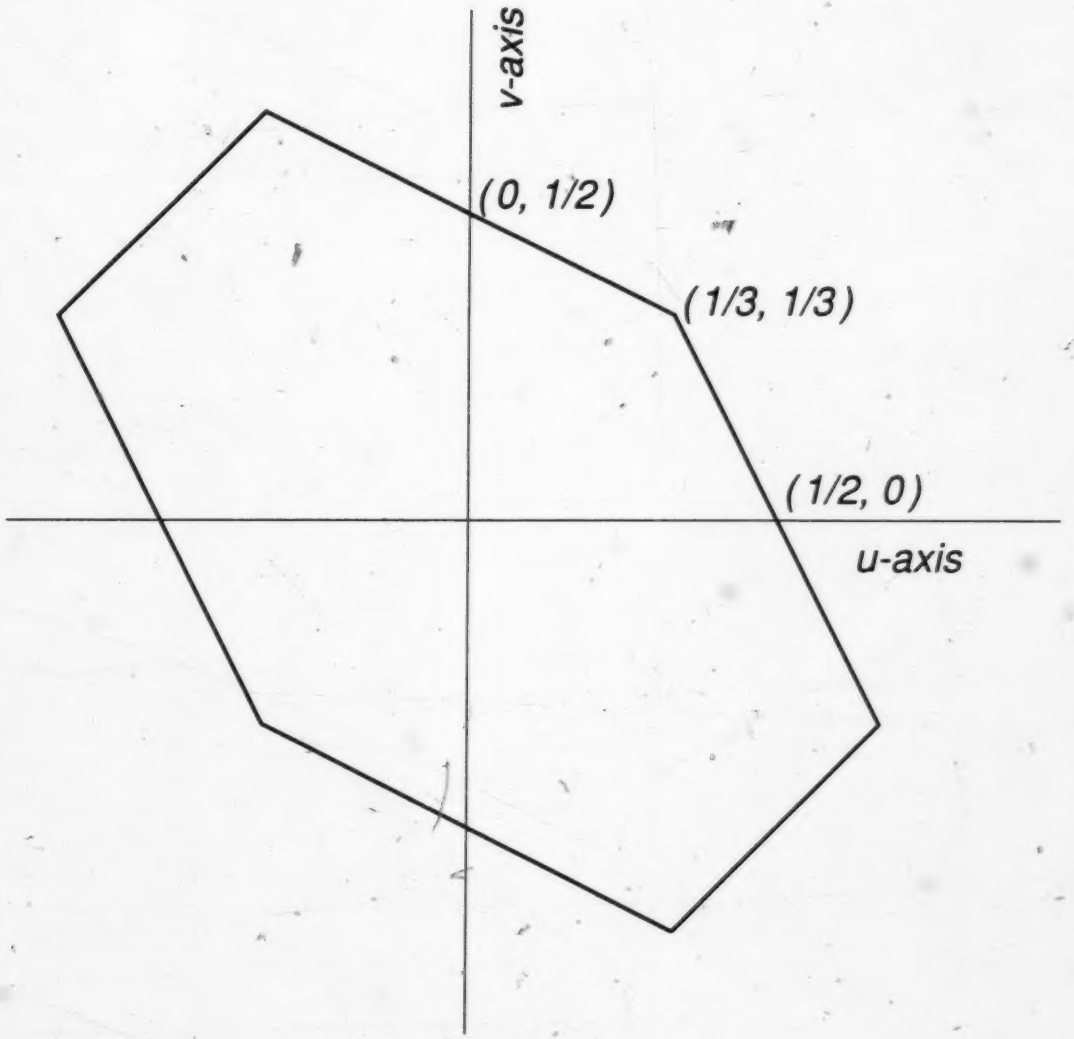

Figure 3.3.I

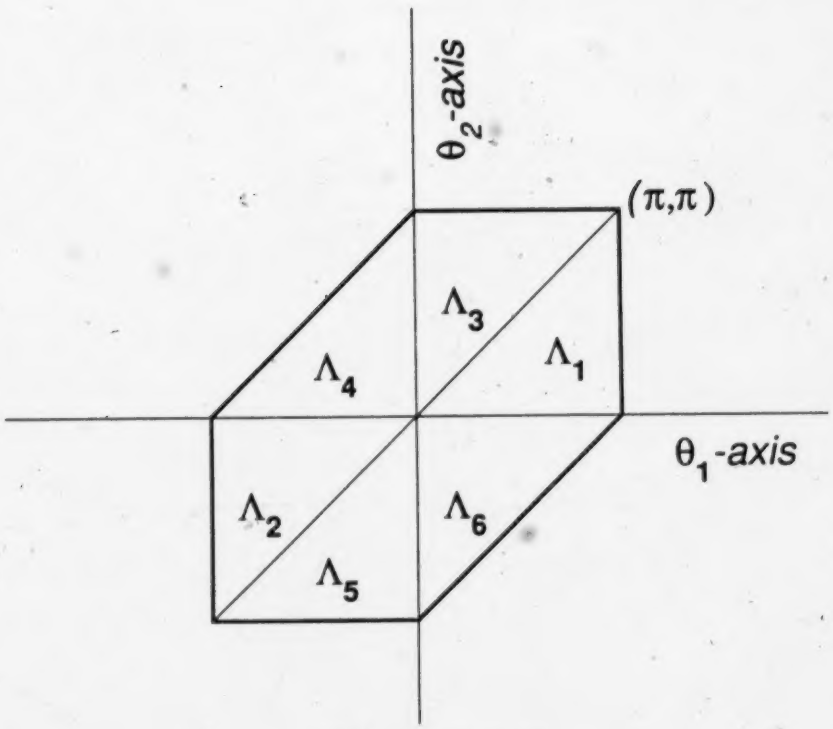

Figure 3.3.II

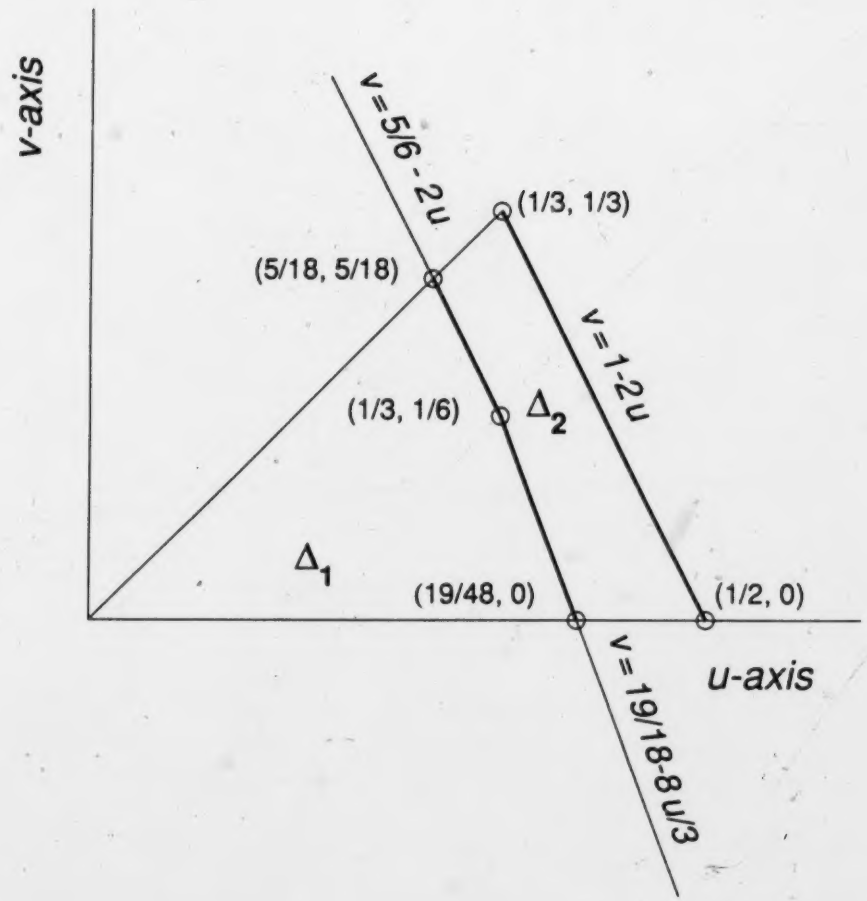

Figure 3.3.III

CHAPTER 4

LINEAR INDEPENDENCE OF BOX SPLINE TRANSLATES

4.1. Introduction

As indicated in the introductory chapter, we now turn to investigate some algebraic properties connected with multi-dimensional box splines. As our first task, we wish to study the linear independence of such box spline translates. Our primary aim in this chapter is to establish a sufficient condition for the linear independence of integer translates of exponential box splines associated with rational matrices. We show that this result is indeed an appropriate extension of the corresponding result known for integer matrices. We also use it to derive a sufficient condition for the linear independence of translates of box splines associated with integer matrices when these translates are restricted to certain submodules of the (multi-) integers.

4.2. A sufficient condition for linear independence

Let $B_{\Xi,\lambda}$ be the exponential box spline associated with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{C}^n$ and an $s \times n$ real matrix Ξ with non-zero columns $x_1, x_2, \ldots, x_n$. Recall that the Fourier transform of $B_{\Xi,\lambda}$ is given by

$$\widehat{B}_{\Xi,\lambda}(x) := \prod_{k=1}^{n} \frac{\sin((i\lambda_k + x_k \cdot x)/2)}{(i\lambda_k + x_k \cdot x)/2}, \qquad y \in \mathbf{C}^s, \tag{4.2.1}$$

and if rank $\Xi = s$, then $B_{\Xi,\lambda}$ is a piecewise exponential polynomial function on $\mathbf{R}^s$.

We first note that as a consequence of (4.2.1), $\theta \in N(B_{\Xi,\lambda})$ (recall (1.4.3)) if and only if for each $l \in \mathbf{Z}^s$, there exists a j, $1 \le j \le n$, such that

$$i\frac{\lambda_j}{2\pi} + (\frac{\theta}{2\pi} + l) \cdot x_j \in \mathbf{Z} \setminus \{0\}. \tag{4.2.2}$$

In particular, if $\lambda = 0$, then

$$N(B_{\Xi,\lambda}) \subseteq \mathbf{R}^s. \tag{4.2.3}$$

Given the matrix Ξ, we define

$$\mathcal{B}(\Xi) := \{Y : \ Y \text{ is an invertible } s \times s \text{ submatrix of } \Xi\},$$

and for $\phi \in \mathbf{C}^s$,

$$\Gamma_\phi(\Xi) := \{x_j : \ \phi \cdot x_j = \lambda_j\}.$$

Further, let

$$\Theta_\lambda(\Xi) := \{\phi \in \mathbf{C}^s : \ \text{span } \Gamma_\phi(\Xi) = \mathbf{R}^s\}.$$

If Ξ is an integer matrix, then the linear independence of the integer translates of $B_{\Xi,\lambda}$ is characterized by the following theorem [R86, Theorem 4.3; DM89, Corollary 4.1].

Theorem 4.2.4. *Let Ξ be an $s \times n$ integer matrix with rank s and non-zero columns and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{C}^n$. Then, the integer translates of $B_{\Xi,\lambda}$ are linearly independent if and only if the following hold:*

(a) $|\det Y| = 1$ for all $Y \in \mathcal{B}(\Xi)$;

(b) $\widehat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ for all $\phi \in \Theta_\lambda(\Xi)$.

Remark 4.2.5. By (4.2.1), $\widehat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ if and only if

$$i\frac{\lambda_k}{2\pi} - i\frac{\phi}{2\pi} \cdot x_k \notin \mathbf{Z} \setminus \{0\}, \quad k = 1, 2, \ldots, n. \tag{4.2.6}$$

Remark 4.2.7. (i) If $n = s = \text{rank } \Xi$, then $\Theta_\lambda(\Xi) = \{\check{\phi}\}$ is a singleton set and $\widehat{B}_{\Xi,\lambda}(-i\check{\phi}) = 1$.

(ii) If $\lambda \in \mathbf{R}^n$, $\Theta_\lambda(\Xi) \subseteq \mathbf{R}^s$, so $\widehat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ for $\phi \in \Theta_\lambda(\Xi)$. So in this case, the linear independence in Theorem 4.2.4 is equivalent to condition (a) which is usually called the **determinant condition**. This condition was first considered in this context by de Boor and Höllig [BH82] who showed it to be necessary for the linear independence of integer translates of polynomial box splines (i.e., $\lambda = 0$) associated with integer matrices. The sufficiency of the condition in this case was subsequently established by Jia [J84] and independently, by Dahmen and Micchelli [DM83]. For arbitrary $\lambda \in \mathbf{R}^n$, the equivalence of linear independence and the determinant condition was first proved by Amos Ron [R88].

In order to deal with rational matrices, we invoke the aid of some general results. Let ψ be a compactly supported continuous function on $\mathbf{R}^s$, and let $K(\psi)$ and $N(\psi)$ be defined, as before, by (1.4.2) and (1.4.3). The following theorem is due to Dahmen and Micchelli [DM83, Theorem 4.1]; its purport, however, also appears to have been known to Lefranc [L58].

Theorem 4.2.8. *Let ψ be a continuous function on $\mathbf{R}^s$ with compact support. Then the integer translates of ψ are linearly independent if and only if there is no $\theta \in \mathbf{C}^s$ such that $(e^{i\theta\cdot j})_{j\in\mathbf{Z}^s} \in K(\psi)$.*

More recently, this fact was formulated by Ron [R89, Theorem 1.1] for compactly supported distributions in terms of their Fourier transforms. His result reads as follows.

Theorem 4.2.9. *Let ψ be a compactly supported distribution on $\mathbf{R}^s$. Then the following hold:*

(a) If $\theta \in \mathbf{C}^s$, then $(e^{i\theta \cdot j})_{j \in \mathbf{Z}^s} \in K(\psi)$ if and only if $\theta \in N(\psi)$.

(b) The integer translates of ψ are linearly independent if and only if the set $N(\psi)$ is empty.

Theorem 4.2.9(b) and 4.2.1 readily imply

Corollary 4.2.10. *Let Ξ be an $s \times n$ real matrix with rank s and non-zero columns $x_1, x_2, \ldots, x_n$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{C}^n$ be given. Then the integer translates of $B_{\Xi,\lambda}$ are linearly independent if and only if for each $\theta \in \mathbf{C}^s$, there exists an $l \in \mathbf{Z}^s$ such that*

$$i\frac{\lambda_k}{2\pi} + (\frac{\theta}{2\pi} + l) \cdot x_k \notin \mathbf{Z} \setminus \{0\} \text{ for all } k = 1, 2, \ldots, n. \tag{4.2.11}$$

With these preliminaries in place, we can now turn to the main results of this section.

Theorem 4.2.12. *Let Ξ be a $1 \times n$ rational matrix with non-zero entries $x_1, x_2, \ldots, x_n$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{C}^n$ and suppose that*

(a) $x_k = \frac{1}{q_k}$, $q_k \in \mathbf{Z} \setminus \{0\}$ for all k;

(b) $\widehat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ for all $\phi \in \Theta_\lambda(\Xi)$.

Then the integer translates of $B_{\Xi,\lambda}$ are linearly independent.

Proof. Let $\theta = 2\pi(\alpha + i\beta) \in \mathbf{C}$ be given and suppose that $\lambda_k = 2\pi(\sigma_k + it_k)$. By (4.2.11), it suffices to find an $l \in \mathbf{Z}$ such that

$$i(\sigma_k + \frac{\beta}{q_k}) + \frac{\alpha + l - q_k t_k}{q_k} \notin \mathbf{Z} \setminus \{0\}, \quad k = 1, 2, \ldots, n. \tag{4.2.13}$$

Assume, without loss of generality,

$$\sigma_k + \frac{\beta}{q_k} = 0, \qquad k = 1, 2, \ldots, n. \tag{4.2.14}$$

Noting that

$$\Theta_\lambda(\Xi) = \left\{ \frac{\lambda_j}{x_j} : \ 1 \le j \le n \right\},$$

we deduce from supposition (b), (4.2.6), and (4.2.14) that for each $j = 1, 2, \ldots, n$,

$$\frac{q_j t_j - q_k t_k}{q_k} \notin \mathbf{Z} \setminus \{0\}, \quad k = 1, 2, \ldots, n. \tag{4.2.15}$$

Now, to prove (4.2.13), consider the following cases.

Case I. $\alpha - q_k t_k \notin \mathbf{Z}$ for all $k = 1, 2, \ldots, n$.

In this case, (4.2.13) holds for $l = 0$.

Case II. There is a j such that $\alpha - q_j t_j \in \mathbf{Z}$.

If so, we choose $l = q_j t_j - \alpha$ and (4.2.13) follows from (4.2.15).

♠

The following example illustrates that condition (b) in Theorem 4.2.12 cannot be omitted.

Example 4.2.16. Let $x_1 = 1$, $x_2 = 1/2$, $\lambda_1 = 2\pi i$, $\lambda_2 = 3\pi i$, so $\Theta_\lambda(\Xi) = \{2\pi i, 6\pi i\}$. Note that $\widehat{B}_{\Xi,\lambda}(6\pi) = 0$ and for $\theta = 2\pi$ and $l \in \mathbf{Z}$,

$$i\frac{\lambda_1}{2\pi} + (\frac{\theta}{2\pi} + l)x_1 = l \tag{4.2.17}$$

and

$$i\frac{\lambda_2}{2\pi} + (\frac{\theta}{2\pi} + l)x_2 = \frac{l-2}{2}. \tag{4.2.18}$$

Now, for each $l \in \mathbf{Z}$, at least one of (4.2.17) or (4.2.18) is a non-zero integer. Hence the integer translates of $B_{\Xi,\lambda}$ are linearly dependent.

Remark 4.2.19. Suppose that Ξ has rank s. If V is an invertible $s \times s$ real matrix and V^T its transpose, then we have the following:

(i) $\phi \in \Theta_\lambda(V\Xi)$ if and only if $V^T\phi \in \Theta_\lambda(\Xi)$.

(ii) It follows from (1.1.2) and (4.2.1) that

$$B_{\Xi,\lambda}(x) = |\det\ V| B_{V\Xi,\lambda}(Vx) \tag{4.2.20}$$

and

$$\widehat{B}_{\Xi,\lambda}(V^T x) = \widehat{B}_{V\Xi,\lambda}(x). \tag{4.2.21}$$

In particular, if V is a *unimodular* (i.e., $|\det\ V|=1$) $s \times s$ *integer matrix*, then $V\mathbf{Z}^s = \mathbf{Z}^s$; so (4.2.20) ensures that the integer translates of $B_{\Xi,\lambda}$ are linearly independent if and only if the integer translates of $B_{V\Xi,\lambda}$ are.

Theorem 4.2.22. *Let Ξ be an $s \times n$ rational matrix with rank s and non-zero columns $x_1, x_2, \ldots, x_n$. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbf{C}^n$ and assume that*

(a) Y^{-1} is an integer matrix for each $Y \in \mathcal{B}(\Xi)$;

(b) $\widehat{B}_{\Xi,\lambda}(-i\phi) \neq 0$ for all $\phi \in \Theta_\lambda(\Xi)$.

Then the integer translates of $B_{\Xi,\lambda}$ are linearly independent.

Proof. The proof proceeds by induction on s and n. Theorem 4.2.12 covers the case $s = 1$. Assume inductively that the result is true for $s - 1$.

Next, the case $n = s$ can be dealt with as follows. Given $\theta \in \mathbf{C}^s$, suppose that $\theta = 2\pi(\alpha + i\beta)$, $\alpha,\ \beta \in \mathbf{R}^s$ and $\lambda_k = 2\pi(\sigma_k + it_k)$, $k = 1, 2, \ldots, s$, with $\sigma_k,\ t_k \in \mathbf{R}$. Solve the system of equations for $l = (l_1, l_2, \ldots, l_s)$:

$$\sum_{\nu=1}^{s} l_\nu x_k^\nu = \lfloor t_k - \alpha \cdot x_k \rfloor, \qquad k = 1, 2, \ldots, s,$$

where $\lfloor a \rfloor$ denotes the integer part of a. Since Ξ^{-1} is now an integer matrix, $l \in \mathbf{Z}^s$ (by Cramer's rule) and (4.2.11) is satisfied for this l. This proves the result for $n = s$ so its validity may be assumed for $n - 1$.

Now, let Ξ be an $s \times n$ matrix. Without loss of generality, we may assume that there exists a $Y \in \mathcal{B}(\Xi)$ containing x_1 as one of its columns and also that the $s \times (n-1)$ matrix with columns $x_2, \ldots, x_n$ has rank s. Let $x_1 = \frac{1}{m} y_1$ where $y_1 \in \mathbf{Z}^s$ and $m \in \mathbf{Z} \setminus \{0\}$. Following the proof of Theorem II.2 in [N72], we can find an $s \times s$ unimodular integer matrix V such that

$$V y_1 = (\tilde{y}_1^1, 0, 0, \ldots, 0)^T, \quad \tilde{y}_1^1 \in \mathbf{Z} \setminus \{0\}.$$

Let $\widetilde{\Xi} := V\Xi$ and denote its columns by $\tilde{x}_i$, $i = 1, \ldots, n$. Note that $\widetilde{\Xi}$ is a rational matrix with non-zero columns and rank s; moreover, it has the form

$$\widetilde{\Xi} = \begin{pmatrix} \tilde{x}_1^1 & \tilde{x}_2^1 & \ldots & \tilde{x}_n^1 \\ 0 & \tilde{x}_2^2 & \ldots & \tilde{x}_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \tilde{x}_2^s & \ldots & \tilde{x}_n^s \end{pmatrix}$$

where $\tilde{x}_1^1 = \frac{\tilde{y}_1^1}{m}$. Furthermore, the $s \times (n-1)$ matrix with columns $\tilde{x}_k$, $k = 2, \ldots, n$, has rank s. In view of Remark 4.2.19(ii), it suffices to prove the theorem for $B_{\widetilde{\Xi},\lambda}$.

If $Z \in \mathcal{B}(\widetilde{\Xi})$, then $Z = VW$ for some $W \in \mathcal{B}(\Xi)$; since W^{-1} and V^{-1} are integer matrices, so is Z^{-1}. This shows that $\widetilde{\Xi}$ satisfies condition (a) of the theorem. By Remark 4.2.19(i) and (4.2.21), $B_{\widetilde{\Xi},\lambda}$ satisfies assumption (b) as well. Therefore, the induction hypotheses can (and will) be applied to submatrices of $\widetilde{\Xi}$.

Let $r_2, r_3, \ldots, r_s$ be integers such that $1 < r_2 < r_3 < \cdots < r_s \leq n$. Note that if the matrix

$$\begin{pmatrix} \tilde{x}_1^1 & \tilde{x}_{r_2}^1 & \cdots & \tilde{x}_{r_s}^1 \\ 0 & \tilde{x}_{r_2}^2 & \cdots & \tilde{x}_{r_s}^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \tilde{x}_{r_2}^s & \cdots & \tilde{x}_{r_s}^s \end{pmatrix} \tag{4.2.23}$$

is invertible, then the inverse, by hypothesis, is an integer matrix. Therefore, $(\tilde{x}_1^1)^{-1} \in \mathbf{Z}$ and the inverse of the matrix

$$\begin{pmatrix} \tilde{x}_{r_2}^2 & \cdots & \tilde{x}_{r_s}^2 \\ \vdots & \ddots & \vdots \\ \tilde{x}_{r_2}^s & \cdots & \tilde{x}_{r_s}^s \end{pmatrix} \tag{4.2.24}$$

exists and is an integer matrix.

Now, let $\theta = (\theta_1, \theta_2, \ldots, \theta_s) \in \mathbf{C}^s$ be given. We will be show that (4.2.11) holds. There are two possible cases.

Case I.

$$\frac{\theta_1}{2\pi} + i\frac{\lambda_1}{2\pi\tilde{x}_1^1} \notin \mathbf{Z}. \tag{4.2.25}$$

Let $\widetilde{\Xi}'$ be the $s \times (n-1)$ submatrix of $\widetilde{\Xi}$ whose columns comprise $\tilde{x}_k$, $k = 2, 3, \ldots, n$. Then $\widetilde{\Xi}'$ satisfies condition (a). Since $B_{\widetilde{\Xi},\lambda}$ satisfies (b) and $\Theta_\lambda(\widetilde{\Xi}') \subseteq \Theta_\lambda(\widetilde{\Xi})$, we infer from (4.2.1) that $B_{\widetilde{\Xi}',\lambda}$ satisfies (b) as well. The induction hypothesis, applied to $B_{\widetilde{\Xi}',\lambda}$, furnishes an $l = (l_1, l_2, \ldots, l_s) \in \mathbf{Z}^s$ such that

$$i\frac{\lambda_k}{2\pi} + (\frac{\theta}{2\pi} + l) \cdot \tilde{x}_k \notin \mathbf{Z} \setminus \{0\}, \quad k = 2, 3, \ldots, n. \tag{4.2.26}$$

Further,

$$i\frac{\lambda_1}{2\pi} + (\frac{\theta}{2\pi} + l) \cdot \tilde{x}_1 = (i\frac{\lambda_1}{2\pi\tilde{x}_1^1} + \frac{\theta_1}{2\pi} + l_1)\tilde{x}_1^1 \notin \mathbf{Z} \tag{4.2.27}$$

as l_1 and $(\tilde{x}_1^1)^{-1}$ are integers and (4.2.25) holds.

This proves (4.2.11) in this case.

Case II.

$$\frac{\theta_1}{2\pi} + i\frac{\lambda_1}{2\pi\tilde{x}_1^1} \in \mathbf{Z}.$$

Rearranging columns $2, 3, \ldots, n$ of $\tilde{\Xi}$ (*and* the corresponding λ_k's), if necessary, assume that for each $k = 2, \ldots, m$, $\tilde{x}_k^j \neq 0$ for some $j = 2, \ldots, s$ and for each $k = m+1, \ldots, n$, $\tilde{x}_k^j = 0$ for all $j = 2, \ldots, s$ and $\tilde{x}_k^1 \neq 0$.

Consider the $(s-1) \times (m-1)$ submatrix $\tilde{\Xi}''$ of $\tilde{\Xi}$ whose columns are $(\tilde{x}_k^2, \ldots, \tilde{x}_k^s)^T$, $k = 2, 3, \ldots, m$. Then $\tilde{\Xi}''$ has non-zero columns and rank $s-1$. If $Z \in \mathcal{B}(\tilde{\Xi}'')$, then it is of the form (4.2.24) and can be extended to an element in $\mathcal{B}(\tilde{\Xi})$ of the form (4.2.23). Consequently, Z^{-1} is an integer matrix. This shows that $\tilde{\Xi}''$ satisfies condition (a). Let $\mu := (\mu_2, \ldots, \mu_m) \in \mathbf{C}^{m-1}$ where

$$\mu_k := \lambda_k - \frac{\lambda_1}{\tilde{x}_1^1}\tilde{x}_k^1, \quad k = 2, 3, \ldots, m. \tag{4.2.28}$$

If

$$\tilde{\phi}'' = (\tilde{\phi}_2'', \ldots, \tilde{\phi}_s'') \in \Theta_\mu(\tilde{\Xi}''),$$

then there exists an $(s-1)\times(s-1)$ matrix in $\mathcal{B}(\tilde{\Xi}'')$ with columns $(\tilde{x}_{r_j}^2, \ldots, \tilde{x}_{r_j}^s)^T$, $j = 2, 3, \ldots, s$, such that

$$\sum_{\nu=2}^{s} \tilde{\phi}_\nu'' \tilde{x}_{r_j}^\nu = \mu_{r_j}, \qquad j = 2, 3, \ldots, s. \tag{4.2.29}$$

Define $\tilde{\phi} := (\frac{\lambda_1}{\tilde{x}_1^1}, \tilde{\phi}_2'', \ldots, \tilde{\phi}_s'') \in \mathbf{C}^s$, so

$$\tilde{\phi} \cdot \tilde{x}_1 = \lambda_1. \tag{4.2.30}$$

Since the $s \times s$ matrix with columns $(\tilde{x}_{r_j})_{j=1}^{s}$ ($r_1 := 1$) belongs to $\mathcal{B}(\widetilde{\Xi})$, (4.2.28), (4.2.29), and (4.2.30) imply that $\tilde{\phi} \in \Theta_\lambda(\widetilde{\Xi})$ and so $\widehat{B}_{\widetilde{\Xi},\lambda}(-i\tilde{\phi}) \neq 0$. Therefore, from (4.2.6) and (4.2.28), $\widehat{B}_{\widetilde{\Xi}'',\mu}(-i\tilde{\phi}'') \neq 0$ showing that $B_{\widetilde{\Xi}'',\mu}$ satisfies condition (b). By the induction hypothesis applied to $B_{\widetilde{\Xi}'',\mu}$, one can find an $(l_2, \ldots l_s) \in \mathbf{Z}^{s-1}$ such that

$$i\frac{\mu_k}{2\pi} + \sum_{\nu=2}^{s}(\frac{\theta_\nu}{2\pi} + l_\nu)\tilde{x}_k^\nu \notin \mathbf{Z} \setminus \{0\}, \quad k = 2, 3, \ldots, m. \tag{4.2.31}$$

Let $l_1 = -\frac{\theta_1}{2\pi} - i\frac{\lambda_1}{2\pi\tilde{x}_1^1}$ so that $l := (l_1, l_2, \ldots, l_s) \in \mathbf{Z}^s$. Clearly,

$$i\frac{\lambda_1}{2\pi} + (\frac{\theta}{2\pi} + l) \cdot \tilde{x}_1 = 0, \tag{4.2.32}$$

and from (4.2.28) and (4.2.31),

$$i\frac{\lambda_k}{2\pi} + (\frac{\theta}{2\pi} + l) \cdot \tilde{x}_k \notin \mathbf{Z} \setminus \{0\}, \quad k = 2, 3, \ldots, m. \tag{4.2.33}$$

Now, for $k = m+1, \ldots, n$, note that

$$\tilde{x}_k = \left(\frac{\tilde{x}_k^1}{\tilde{x}_1^1}\right)\tilde{x}_1 \tag{4.2.34}$$

and

$$i\frac{\lambda_k}{2\pi} + (\frac{\theta}{2\pi} + l) \cdot \tilde{x}_k = i\frac{\lambda_k}{2\pi} - i\frac{\lambda_1}{2\pi\tilde{x}_1^1}\tilde{x}_k^1. \tag{4.2.35}$$

Choose $\varrho \in \Theta_\lambda(\widetilde{\Xi})$ so that

$$\varrho \cdot \tilde{x}_1 = \lambda_1. \tag{4.2.36}$$

As $\widehat{B}_{\widetilde{\Xi},\lambda}(-i\varrho) \neq 0$, it follows from (4.2.6), (4.2.34), and (4.2.36) that

$$i\frac{\lambda_k}{2\pi} - i\frac{\lambda_1}{2\pi\tilde{x}_1^1}\tilde{x}_k^1 \notin \mathbf{Z} \setminus \{0\}, \quad k = m+1, \ldots, n. \tag{4.2.37}$$

From (4.2.37), (4.2.35), (4.2.33), and (4.2.32), we conclude that (4.2.11) holds once again and this completes the proof.

♠

Remark 4.2.38. In comparing Theorem 4.2.22 with Theorem 4.2.4, note that if Ξ is an integer matrix and $Y \in \mathcal{B}(\Xi)$, Y^{-1} is an integer matrix if and only if $|\det Y|=1$. This shows that Theorem 4.2.22 is an appropriate extension of Theorem 4.2.4 to rational matrices.

Remark 4.2.39. In general, condition (a) of Theorem 4.2.22 is not necessary for linear independence. As a simple example, consider

$$\Xi = \begin{pmatrix} 3/2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \lambda = 0.$$

Obviously, Ξ^{-1} is not an integer matrix but for any $\theta = (\theta_1, \theta_2) \in \mathbf{R}^2$, $0 \leq \theta_i < 2\pi$, there is an $(l_1, 0) \in \mathbf{Z}^2$ such that (4.2.11) holds. This shows (by (4.2.3)) that $N(B_\Xi)/2\pi\mathbf{Z}^2$ and hence $N(B_\Xi)$ is an empty set. That is, the integer translates of B_Ξ are linearly independent. (For more relevant discussion in this regard, see [JS89, Section 3].)

Remark 4.2.40. If $\lambda \in \mathbf{R}^n$, then Remark 4.2.7(i) and Theorem 4.2.22 show that the integer translates of $B_{\Xi,\lambda}$ are linearly independent provided Y^{-1} is an integer matrix for every $Y \in \mathcal{B}(\Xi)$. In particular, if $\lambda = 0$, then one recovers [JS89, Theorem 2.4].

The next result is an application of Theorem 4.2.22 to polynomial box splines associated with integer matrices.

Corollary 4.2.41. *Let Ξ be an $s \times n$ integer matrix with rank s. Suppose that A is an invertible $s \times s$ integer matrix such that the submodule*

$$\mathcal{A} := A\mathbf{Z}^s \subseteq X\mathbf{Z}^s \quad \text{for all } X \in \mathcal{B}(\Xi).$$

Then the $\mathcal{A}$-translates of (the polynomial box spline) B_Ξ are linearly independent.

Proof. Firstly, it follows quite easily from (4.2.20) (with $\lambda = 0$) that the $\mathcal{A}$-translates of B_Ξ are linearly independent if and only if the $\mathbf{Z}^s$-translates of $B_{A^{-1}\Xi}$ are.

Now, note that $A^{-1}\Xi$ is a rational matrix and if $Y \in \mathcal{B}(A^{-1}\Xi)$, then $Y = A^{-1}X$ for some $X \in \mathcal{B}(\Xi)$. Since $A\mathbf{Z}^s \subseteq X\mathbf{Z}^s$, there exists an $s \times s$ integer matrix U such that $A = XU$. Consequently, $Y^{-1} = U$ is an integer matrix and the desired result follows from Remark 4.2.40.

♠

We conclude this chapter with the following development prompted by a study of [DJM89-2]. Its purpose is to reformulate condition (b) of Theorem 4.2.22. Similar considerations will figure prominently in the next chapter as well.

Suppose that Y is an $s \times k$ submatrix of Ξ . We denote by λ_Y the vector in $\mathbf{C}^k$ whose components are those λ_j's corresponding to the columns of Y. We employ the notation $y \in Y$ to denote that y is a column of Y and use $(\Xi \setminus Y)$ to denote the $s \times (n-k)$ submatrix of Ξ whose columns consist of those $\xi \in \Xi$

which are not columns of Y. It must be noted that Y and $(\Xi \setminus Y)$ *can* have columns in common. Further, we let

$$\langle Y \rangle := \left\{ \sum a_y y \ : \ a_y \in \mathbf{R}, \ y \in Y \right\}.$$

With these notations in place, it is not hard to see using (4.2.6) that condition (b) of Theorem 4.2.22 is in fact equivalent to the following:

(b') *Let Y be any element in $\mathcal{B}(\Xi)$. For $\xi \in (\Xi \setminus Y)$,*

$$\xi = \sum_{y \in Y} a_y y \quad \Rightarrow \quad \lambda_\xi - \sum_{y \in Y} a_y \lambda_y \notin 2\pi i \mathbf{Z} \setminus \{0\}.$$

CHAPTER 5

BOX SPLINES WITH FINITE DIMENSIONAL KERNELS

5.1. Introduction

Let $B_{\Xi,\lambda}$ be the exponential box spline associated with an $s \times n$ rational matrix Ξ and $\lambda \in \mathbf{C}^n$. Having discussed, in the previous chapter, conditions under which the associated kernel space $K(B_{\Xi,\lambda})$ is trivial, we now take up the question of its finite dimensionality. The goal of this chapter is to establish a sufficient condition for the latter phenomenon to hold. Once again, this result extends the known theorem for integer matrices.

5.2. On a property of matrices

The purpose of this section is to lay down some preparatory groundwork needed in the subsequent sections. We recall the following definition from [DJM88].

Definition 5.2.1. An $s \times n$ *integer matrix* is called **weakly unimodular** if for each $s \times k$ $(k \leq s-1)$ submatrix W of Ξ with full rank (i.e., the columns of W are linearly independent), the greatest common divisor (g.c.d) of all its $k \times k$ minors is 1.

The following proposition is stated in [DJM88, Remark 1.1]. We include a proof here for sake of completeness.

Proposition 5.2.2. *Let Ξ be an $s \times n$ integer matrix. Then Ξ is weakly unimodular if and only if every $s \times k$ $(k \leq s-1)$ submatrix of Ξ with full rank can be completed to an $s \times s$ unimodular integer matrix.*

Proof. To show sufficiency, let W be any $s \times k$ $(k \leq s-1)$ submatrix of Ξ with full rank. Then there exists an $s \times (s-k)$ integer matrix W_1 such that the completed matrix $\widetilde{W} = [W \ W_1]$ is unimodular. As $(\widetilde{W})^{-1}$ has only integer entries and

$$(\widetilde{W})^{-1}[W \ W_1] = I_s,$$

the $s \times s$ identity matrix, it follows from the Cauchy-Binet theorem (see for instance [N72, p. 25]) that there exist integers λ_j (which are in fact minors of $(\widetilde{W})^{-1}$) such that $\sum \lambda_j m_j = 1$ where $\{m_j\}$ is the set of all $k \times k$ minors of W. This shows that g.c.d (m_j)=1 as desired.

The reverse implication is a consequence of the more general fact (see [N72, p. 38]) that any $s \times k$ $(k \leq s-1)$ integer matrix W of full rank can be completed to an $s \times s$ integer matrix the absolute value of whose determinant equals that of the greatest common divisor of all the $k \times k$ minors of W.

♠

Motivated by Definition 5.2.1 and the preceding proposition, we introduce the following

Definition 5.2.3. An $s \times n$ *rational matrix* Ξ is said to have **property** $\mathcal{E}$ (extendibility) if every $s \times k$ $(k \leq s-1)$ submatrix W of Ξ with full rank can be completed to an invertible $s \times s$ rational matrix whose inverse has only integer entries. (Note that it actually suffices to check the extendibility criterion only for $k = \min\{s-1,\ rank\ \Xi\}$.)

Remark 5.2.4. If V is a unimodular $s \times s$ integer matrix, then it is not hard to see that Ξ has property $\mathcal{E}$ if and only if $V\Xi$ has the same property.

We now show that property $\mathcal{E}$ is an appropriate extension of the notion of weak unimodularity to rational matrices.

Proposition 5.2.5. *Let Ξ be an $s \times n$ integer matrix. Then, Ξ has property $\mathcal{E}$ if and only if it is weakly unimodular.*

Proof. If Ξ is weakly unimodular, it has property $\mathcal{E}$ by Proposition 5.2.2. The converse statement is proved in the same manner as in Proposition 5.2.2 by using the Cauchy-Binet theorem.

♠

We observe in passing that *every* $1 \times n$ rational matrix possesses property $\mathcal{E}$ (vacuously); the following result characterizes property $\mathcal{E}$ when $s = 2$.

Proposition 5.2.6. *Let Ξ be a $2 \times n$ rational matrix with non-zero columns $x_1, \ldots, x_n$. Let $x_i = \frac{1}{q_i}(p_i^1, p_i^2)^T$, $i = 1, 2, \ldots, n$, where $p_i^1, p_i^2, q_i \in \mathbf{Z}$. Then Ξ has property $\mathcal{E}$ if and only if g.c.d (p_i^1, p_i^2) divides q_i for every $1 \leq i \leq n$.*

Proof. Suppose Ξ has property $\mathcal{E}$. Then each column x_i can be completed to a 2×2 rational matrix whose inverse is an integer matrix. Consequently, there exist integers a_i^1, a_i^2 such that $a_i^1 p_i^1 + a_i^2 p_i^2 = q_i$; ergo, g.c.d (p_i^1, p_i^2) divides q_i.

On the other hand, suppose that g.c.d (p_i^1, p_i^2) divides q_i and choose b_i^1, $b_i^2 \in \mathbf{Z}$ such that $p_i^1 b_i^2 - p_i^2 b_i^1 =$ g.c.d (p_i^1, p_i^2). Then the matrix

$$\frac{1}{q_i}\begin{pmatrix} p_i^1 & b_i^1 \\ p_i^2 & b_i^2 \end{pmatrix}$$

is a completion of x_i and it is rather easy to see that its inverse has only integer entries. This being true for every i, $1 \leq i \leq n$, we conclude that Ξ has property $\mathcal{E}$.

♠

In general, if W is an $s \times k$ $(k \leq s-1)$ rational matrix of full rank, then $W = \frac{1}{q(W)} W'$ where $q(W) \in \mathbf{Z} \setminus \{0\}$ and W' is an $s \times k$ integer matrix of full rank. As noted earlier, W' can be completed to an $s \times s$ integer matrix $\widetilde{W'}$ such that $|\det \widetilde{W'}| = d_k(W) :=$ g.c.d ($k \times k$ minors of W'). So an $s \times n$ rational matrix Ξ has property $\mathcal{E}$ if for each $s \times k$ $(k \leq s-1)$ submatrix W of Ξ with full rank, $d_k(W)$ divides $q(W)$. (If so, then the matrix $\widetilde{W} = \frac{1}{q(W)} \widetilde{W'}$ is a completion of W and $(\widetilde{W})^{-1}$ has only integer entries.)

We end this section with a simple result which will be used later.

Proposition 5.2.7. *Let A be an $s \times n$ rational matrix with non-zero columns and of the form*

$$A = \begin{pmatrix} a_1^1 & a_2^1 & \dots & a_n^1 \\ 0 & a_2^2 & \dots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_2^s & \dots & a_n^s \end{pmatrix}.$$

Define A_1 to be the $s \times (n-1)$ submatrix of A whose columns comprise $(a_j^1, \dots, a_j^s)^T$, $j = 2, 3, \dots, n$, and let A_2 be the $(s-1) \times (n-1)$ submatrix of A with columns $(a_j^2, \dots, a_j^s)^T$, $j = 2, 3, \dots, n$. Assume that A has property $\mathcal{E}$. Then,

(i) A_1 and A_2 both have property $\mathcal{E}$;

(ii) $(a_1^1)^{-1} \in \mathbf{Z}$.

Proof. (i) That A_1 has property $\mathcal{E}$ is quite evident. Turning to A_2, let W be any $(s-1) \times k$ $(k \leq s-2)$ submatrix of A_2 with full rank, say,

$$W = \begin{pmatrix} a_{r_1}^2 & \dots & a_{r_k}^2 \\ \vdots & \ddots & \vdots \\ a_{r_1}^s & \dots & a_{r_k}^s \end{pmatrix}.$$

The $s \times (k+1)$ matrix W_0 with columns $(a_1^1, 0, \dots, 0)^T$, $(a_{r_j}^1, \dots, a_{r_j}^s)^T$, $j = 1, \dots, k$ has full rank and since A has property $\mathcal{E}$, W_0 can be completed to an $s \times s$ rational matrix $\widetilde{W_0}$ whose inverse is an integer matrix. Let

$$\widetilde{W_0} = \begin{pmatrix} a_1^1 & a_{r_1}^1 & \dots & a_{r_k}^1 & b_1^1 & \dots & b_{s-k-1}^1 \\ 0 & a_{r_1}^2 & \dots & a_{r_k}^2 & b_1^2 & \dots & b_{s-k-1}^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_{r_1}^s & \dots & a_{r_k}^s & b_1^s & \dots & b_{s-k-1}^s \end{pmatrix}.$$

The $(s-1) \times (s-1)$ matrix $\widetilde{W}$ with columns $(a_{r_j}^2, \dots, a_{r_j}^s)^T$, $j = 1, 2, \dots, k$, $(b_j^2, \dots, b_j^s)^T$, $j = 1, 2, \dots, s-k-1$, is invertible and its inverse has only integer entries. This shows that A_2 has property $\mathcal{E}$.

(ii) There exists an $s \times s$ rational matrix Y with first column $(a_1^1, 0, \ldots, 0)^T$ and whose inverse is an integer matrix. It follows that $(a_1^1)^{-1} \in \mathbf{Z}$.

♠

5.3. Upon the dimensionality of $K(B_{\Xi,\lambda})$

The finite dimensionality of $K(B_{\Xi,\lambda})$ was first considered, albeit implicitly, by Dahmen and Micchelli [DM85] in the case of polynomial box splines (i.e., $\lambda = 0$) associated with integer matrices. The following is a thinly veiled version of their result; the sufficiency part of its proof is contained in Theorem 4.2 of [DM85] whereas the necessity is implicit in the remarks following Example 4.1 there.

Theorem 5.3.1. *Let Ξ be an $s \times n$ integer matrix with rank s and non-zero columns. Then $K(B_\Xi)$ is finite dimensional if and only if for each $s \times k$ submatrix Y of Ξ with rank $Y < s$, the translates $\{B_Y(\cdot - j) : j \in \mathbf{Z}^s\}$ are linearly independent.*

Although this result does characterize the finite dimensionality of $K(B_\Xi)$ completely, the characterizing condition is not given explicitly in terms of the matrix Ξ. Such an explicit characterization was recently provided by Dahmen, Jia, and Micchelli [DJM88] who did so by first establishing the following general result [DJM88, Theorem 2.1].

Theorem 5.3.2. *Let ψ be a compactly supported distribution on $\mathbf{R}^s$. Then $K(\psi)$ is finite dimensional if and only if $N(\psi)/2\pi\mathbf{Z}^s$ is a finite set.*

Dahmen *et al*(loc.cit.,) used this theorem along with Theorem 5.3.1 to obtain

Theorem 5.3.3. *Let Ξ be an $s \times n$ integer matrix with rank s and non-zero columns. Then, $K(B_\Xi)$ is finite dimensional if and only if Ξ is weakly unimodular.*

Subsequent to this, Dahmen, Jia, and Micchelli continued their investigations [DJM89-1, DJM89-2] by considering the finite dimensionality of $K(B_{\Xi,\lambda})$ for integer matrices Ξ and $\lambda \in \mathbf{C}^n$. Once again, they were aided in their efforts by Theorem 5.3.2 and the following analogue of Theorem 5.3.1. The notation used in its statement should be recalled from the previous chapter.

Theorem 5.3.4. *Let Ξ be an $s \times n$ integer matrix with rank s and non-zero columns and suppose that $\lambda \in \mathbf{C}^n$ is given. Then $K(B_{\Xi,\lambda})$ is finite dimensional if and only if for each $s \times k$ submatrix Y of Ξ with rank $Y < s$, the translates $\{B_{Y,\lambda_Y}(\cdot - j) \ : \ j \in \mathbf{Z}^s\}$ are linearly independent.*

The proof of the result above involves appropriate modifications of that of Theorem 5.3.1 and as such, does not appear anywhere. However, these modifications are easy to make because the arguments in question rely only on certain common features shared by both polynomial and exponential box splines associated with integer matrices.

The next result, proved in [DJM89-2], supplies a complete characterization for the finite dimensionality of $K(B_{\Xi,\lambda})$ for integer matrices. Once again, we use some notation that was introduced towards the end of the last chapter.

Theorem 5.3.5. *Let Ξ be an $s \times n$ integer matrix with non-zero columns $x_1, \ldots, x_n$ and rank s. Suppose that $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ is given. Then, $K(B_{\Xi,\lambda})$ is finite dimensional if and only if the following conditions hold:*

(a) Ξ is weakly unimodular;

(b) Let Y be any $s \times (s-1)$ submatrix of Ξ with full rank (i.e., the columns of Y are linearly independent). For $\xi \in (\Xi \setminus Y) \cap \langle Y \rangle$,

$$\xi = \sum_{y \in Y} a_y y \quad \Rightarrow \quad \lambda_\xi - \sum_{y \in Y} a_y \lambda_y \notin 2\pi i \mathbf{Z} \setminus \{0\}.$$

Remark 5.3.6. If Ξ is an $s \times n$ matrix and $\lambda \in \mathbf{C}^n$, then $B_{\Xi,\lambda}$ satisfies condition (b) of Theorem 5.3.5 if and only if $B_{V\Xi,\lambda}$ satisfies the same condition for any $s \times s$ invertible matrix V.

Condition (b) of Theorem 5.3.5 should be compared with condition (b') given at the end of Chapter 4. The former can be checked quite easily using the following procedure outlined in [DJM89-2].

Suppose that Ξ is an $s \times n$ real matrix and Y an $s \times k$ submatrix of Ξ with linearly independent columns, x_{r_j}, $j = 1, \ldots, k$. Assume that

$$\xi = (\xi^1, \ldots, \xi^s) \in (\Xi \setminus Y) \cap \langle Y \rangle$$

with

$$\xi = \sum_{j=1}^{k} a_j x_{r_j}.$$

There is a $k \times k$ invertible submatrix Y' of Y with columns say, $(x_{r_j}^{l_1}, \ldots, x_{r_j}^{l_k})^T$, $j = 1, \ldots, k$, such that

$$\sum_{j=1}^{k} a_j x_{r_j}^{l_p} = \xi^{l_p}, \quad p = 1, \ldots, k.$$

Defining

$$\widetilde{Y'} := \begin{pmatrix} x_{r_1}^{l_1} & \dots & x_{r_k}^{l_1} & \xi^{l_1} \\ \vdots & \ddots & \vdots & \vdots \\ x_{r_1}^{l_k} & \dots & x_{r_k}^{l_k} & \xi^{l_k} \\ \lambda_{r_1} & \dots & \lambda_{r_k} & \lambda_\xi \end{pmatrix},$$

it is not hard to see that

$$\lambda_\xi - \sum_{j=1}^{k} a_j \lambda_{r_j} = \frac{\det \widetilde{Y'}}{\det Y'}.$$

Our objective now is to generalize the sufficiency part of Theorem 5.3.5 to rational matrices. We begin with the one-dimensional case. The following theorem is a special case of [R89, Corollary 2.4(a)], but we provide a self-contained (and different) proof for sake of completeness.

Theorem 5.3.7. *Let Ξ be a $1 \times n$ rational matrix with non-zero columns $x_1, \dots, x_n$ and let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n$. Then $K(B_{\Xi,\lambda})$ is always finite dimensional.*

Proof. In view of Theorem 5.3.2, it suffices to show that the set

$$S := \{\theta = 2\pi(\alpha + i\beta) \in N(B_{\Xi,\lambda}) : \ \alpha, \beta \in \mathbf{R}, \ 0 \le \alpha < 1\}$$

is finite. We let $x_k = \frac{p_k}{q_k}$, $k = 1, \dots, n$, where p_k, $q_k \in \mathbf{Z}$ and $p_k > 0$. Let us suppose that $\lambda_k = 2\pi(\sigma_k + it_k)$, $k = 1, \dots, n$, σ_k, $t_k \in \mathbf{R}$, and define

$$R_k := \left\{ \frac{j + t_k q_k}{p_k} : \ j \in \mathbf{Z}, \quad -t_k q_k \le j < p_k - t_k q_k \right\}.$$

Further, let

$$\mathcal{R} := \bigcup_{k=1}^{n} R_k$$

and

$$\mathcal{I} := \left\{ \frac{-\sigma_k q_k}{p_k} : \ 1 \le k \le n \right\}.$$

Now let $\theta = 2\pi(\alpha + i\beta) \in S$. By (4.2.2), there exists a k, $1 \le k \le n$, such that

$$i(\sigma_k + \frac{\beta p_k}{q_k}) + (\alpha - \frac{t_k q_k}{p_k})\frac{p_k}{q_k} \in \mathbf{Z} \setminus \{0\}.$$

It follows that

$$\sigma_k + \frac{\beta p_k}{q_k} = 0 \tag{5.3.8}$$

and

$$\alpha p_k - t_k q_k \in \mathbf{Z}. \tag{5.3.9}$$

From (5.3.8), (5.3.9), and the fact that $0 \le \alpha < 1$, we conclude that

$$\alpha \in \mathcal{R} \quad \text{and} \quad \beta \in \mathcal{I}.$$

This finishes the proof because both $\mathcal{R}$ and $\mathcal{I}$ are finite sets.

♠

Remark 5.3.10. If V is a unimodular $s \times s$ integer matrix, then so is V^{-1}. Therefore, by (4.2.20), the sequence $\left(a(j)\right)_{j \in \mathbf{Z}^s} \in K(B_{\Xi,\lambda})$ if and only if $\left(a(V^{-1}j)\right)_{j \in \mathbf{Z}^s} \in K(B_{V\Xi,\lambda})$. Consequently, either one of these spaces is finite dimensional precisely when the other is.

We now pass to higher dimensions. In what follows, if $\phi = (\phi_1, \phi_2, \ldots, \phi_s) \in \mathbf{C}^s$, then we denote its projection $(\phi_2, \ldots, \phi_s) \in \mathbf{C}^{s-1}$ by $P(\phi)$.

Theorem 5.3.11. *Let Ξ be an $s \times n$ rational matrix with non-zero columns $x_1, \ldots, x_n$ and rank s. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ be given and suppose that the following two conditions hold:*

(a) Ξ has property $\mathcal{E}$;

(b) Let Y be any $s \times (s-1)$ submatrix of Ξ with full rank. For $\xi \in (\Xi \setminus Y) \cap \langle Y \rangle$,

$$\xi = \sum_{y \in Y} a_y y \quad \Rightarrow \quad \lambda_\xi - \sum_{y \in Y} a_y \lambda_y \notin 2\pi i \mathbf{Z} \setminus \{0\}.$$

Then $K(B_{\Xi,\lambda})$ is finite dimensional.

Proof. We shall use induction on s and n. Theorem 5.3.7 covers the case $s = 1$. We may therefore assume the validity of the result for $s - 1$.

Now for the case $n = s$. It may be assumed that Ξ is of the form

$$\Xi = \begin{pmatrix} x_1^1 & x_2^1 & \ldots & x_s^1 \\ 0 & x_2^2 & \ldots & x_s^2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2^s & \ldots & x_s^s \end{pmatrix}.$$

(Otherwise, we can find a unimodular $s \times s$ integer matrix V such that $V\Xi$ has the above form. This matrix also satisfies conditions (a) [Remark 5.2.4] and (b) [Remark 5.3.6] and it suffices to prove that $K(B_{V\Xi,\lambda})$ is finite dimensional [Remark 5.3.10]).

Note that Proposition 5.2.7(ii) guarantees that

$$(x_1^1)^{-1} \in \mathbf{Z}. \tag{5.3.12}$$

We need to show (by Theorem 5.3.2) that $N(B_{\Xi,\lambda})/2\pi\mathbf{Z}^s$ is a finite set. Let $\theta = (\theta_1, \ldots, \theta_s) \in N(B_{\Xi,\lambda})$, $\theta_j = 2\pi(\alpha_j + i\beta_j)$, $\alpha_j,\ \beta_j \in \mathbf{R}$ and $0 \le \alpha_j < 1$. Also suppose that $\lambda_k = 2\pi(\sigma_k + it_k)$, $\sigma_k,\ t_k \in \mathbf{R}$, $k = 1, \ldots, s$.

Let Ξ' be the $s \times (s-1)$ matrix with columns x_k, $k = 2, \ldots, s$, and $\lambda' := P(\lambda)$. Since Ξ has property $\mathcal{E}$, there exists $z = (z^1, \ldots, z^s) \in \mathbf{R}^s$ such that the $s \times s$ matrix $\widetilde{\Xi}'$ with columns x_k, $k = 2, \ldots, s$, z (in that order), has an inverse all of whose entries are integers. Defining $\tilde{\lambda}' := (\lambda_2, \ldots, \lambda_s, 0)$, it follows from Remark 4.2.7(ii) and Theorem 4.2.22 that $N(B_{\widetilde{\Xi}', \tilde{\lambda}'})$ is empty. As $N(B_{\Xi', \lambda'}) \subseteq N(B_{\widetilde{\Xi}', \tilde{\lambda}'})$ and $\theta \in N(B_{\Xi, \lambda})$, there exists a $d = (d_1, \ldots, d_s) \in \mathbf{Z}^s$ such that

$$i\frac{\lambda_1}{2\pi} + (\frac{\theta}{2\pi} + d) \cdot x_1 \in \mathbf{Z} \setminus \{0\}. \tag{5.3.13}$$

As $(x_1^1)^{-1}$ and d_1 are integers, we have from (5.3.13),

$$\sigma_1 + \beta_1 x_1^1 = 0 \tag{5.3.14}$$

and

$$\alpha_1 - \frac{t_1}{x_1^1} \in \mathbf{Z}. \tag{5.3.15}$$

Conditions (5.3.14), (5.3.15), and the fact that $0 \leq \alpha_1 < 1$ fix θ_1 uniquely. Also note that (5.3.12) and (5.3.13) imply

$$i\frac{\lambda_1}{2\pi x_1^1} + \frac{\theta_1}{2\pi} \in \mathbf{Z}. \tag{5.3.16}$$

Now, let Ξ'' be the $(s-1) \times (s-1)$ submatrix of Ξ with non-zero columns $P(x_k)$, $k = 2, \ldots, s$, and rank $s-1$. Define $\mu := (\mu_2, \ldots, \mu_s) \in \mathbf{C}^{s-1}$ where

$$\mu_k := \lambda_k - \frac{\lambda_1}{x_1^1} x_k^1, \quad k = 2, \ldots, s. \tag{5.3.17}$$

The matrix Ξ'' has property $\mathcal{E}$ by Proposition 5.2.7(i) and $B_{\Xi'',\mu}$ satisfies condition (b) vacuously because Ξ'' has rank $s-1$. Let $(l_2,\ldots,l_s)\in \mathbf{Z}^{s-1}$ and set

$$l_1 := -i\frac{\lambda_1}{2\pi x_1^1} - \frac{\theta_1}{2\pi}.$$

For $l=(l_1,l_2,\ldots,l_s)\in\mathbf{Z}^s$, there is a k (by (4.2.2)) such that $1\le k\le s$ and

$$i\frac{\lambda_k}{2\pi}+(\frac{\theta}{2\pi}+l)\cdot x_k \in \mathbf{Z}\setminus\{0\}. \tag{5.3.18}$$

By the choice of l_1, $2\le k\le s$; thus, from (5.3.18), (5.3.17), and the fact that $l_2,\ldots,l_s$ are arbitrary, we conclude that

$$P(\theta)\in N(B_{\Xi'',\mu})/2\pi\mathbf{Z}^{s-1}$$

which is a finite set by the induction hypothesis.

This completes the proof for the case $n=s$. Assume now that the theorem is true for $n-1$.

Although the proof of the general case is quite similar to that of the previous one, there are some notable differences that ought to be dealt with in some detail.

To proceed, we may once again assume that Ξ is of the form

$$\Xi=\begin{pmatrix} x_1^1 & x_2^1 & \ldots & x_n^1\\ 0 & x_2^2 & \ldots & x_n^2\\ \vdots & \vdots & \ddots & \vdots\\ 0 & x_2^s & \ldots & x_n^s\end{pmatrix}$$

with $(x_1^1)^{-1}\in\mathbf{Z}$. We may also assume that the $s\times(n-1)$ matrix with columns x_k, $k=2,\ldots,n$, has rank s. By rearranging, if need be, columns 2 through

n (*and* the corresponding λ_k's) we may assume that for each $k = 2, \ldots, m$, $P(x_k) \neq 0$ whereas for $k = m+1, \ldots, n$, $P(x_k) = 0$. We designate Ξ' to be the $s \times (n-1)$ submatrix of Ξ with columns x_k, $k = 2, \ldots, n$, and $\lambda' = (\lambda_2, \ldots, \lambda_n)$. Note that Ξ' has property $\mathcal{E}$ [Proposition 5.2.7(ii)] and $B_{\Xi',\lambda'}$ satisfies condition (b) because $B_{\Xi,\lambda}$ does. By the induction hypothesis applied to $B_{\Xi',\lambda'}$, it suffices to show that the set $\big(N(B_{\Xi,\lambda}) \setminus N(B_{\Xi',\lambda'})\big)/2\pi\mathbf{Z}^s$ is finite. If $\theta = (\theta_1, \ldots, \theta_s)$ belongs to this set, there must be a $d = (d_1, \ldots, d_s) \in \mathbf{Z}^s$ for which (5.3.13) holds. Once again this fixes θ_1 uniquely and (5.3.16) holds as well. Choosing integers $l_2, \ldots, l_s$ arbitrarily and l_1 as before, we conclude that (5.3.18) holds for some k, $2 \leq k \leq n$. Suppose now that $k \in \{m+1, \ldots, n\}$. We can find an $s \times (s-1)$ submatrix Y of Ξ with full rank and whose first column is x_1. Clearly,

$$x_k \in (\Xi \setminus Y) \cap \langle Y \rangle \text{ and } x_k = \frac{x_k^1}{x_1^1} x_1. \tag{5.3.19}$$

Further,

$$i\frac{\lambda_k}{2\pi} + (\frac{\theta}{2\pi} + l) \cdot x_k = i\frac{\lambda_k}{2\pi} - i\frac{\lambda_1}{2\pi x_1^1} x_k^1. \tag{5.3.20}$$

From (5.3.19), condition (b), and (5.3.20), we conclude that (5.3.18) does not hold for any $k \in \{m+1, \ldots, n\}$. It must therefore hold for some $k \in \{2, \ldots, m\}$.

We now pass to the $(s-1) \times (m-1)$ matrix Ξ'' whose columns comprise $P(x_k)$, $k = 2, \ldots, m$, and define $\mu := (\mu_2, \ldots, \mu_m) \in \mathbf{C}^{m-1}$ where μ_k is defined by (5.3.17) for each $k = 2, \ldots, m$. The preceding arguments have in effect shown that

$$P(\theta) \in N(B_{\Xi'',\mu})/2\pi\mathbf{Z}^{s-1}.$$

In order to apply the induction hypothesis to $B_{\Xi'',\mu}$ and thereby complete the proof, it remains to demonstrate that $B_{\Xi'',\mu}$ satisfies the conditions of the theorem.

To that end, we first note that Ξ'' has non-zero columns and rank $s-1$ and it satisfies condition (a) by Proposition 5.2.7(i). As for condition (b), let Y'' be an $(s-1)\times(s-2)$ submatrix of Ξ'' with linearly independent columns $P(x_{r_k})$, $k=2,\ldots,s-1$. Assume that $P(\xi)\in(\Xi''\setminus Y'')\cap\langle Y''\rangle$, $\xi=(\xi^1,\ldots,\xi^s)\in\Xi$, and

$$P(\xi)=\sum_{k=2}^{s-1}a_kP(x_{r_k}). \tag{5.3.21}$$

Define $r_1:=1$ and choose $a_1\in\mathbf{R}$ such that

$$\sum_{k=1}^{s-1}a_kx^1_{r_k}=\xi^1. \tag{5.3.22}$$

The $s\times(s-1)$ submatrix Y of Ξ with columns x_{r_k}, $k=1,\ldots,s-1$, is of full rank and it is clear that $\xi\in(\Xi\setminus Y)$. Further, it follows from (5.3.21) and (5.3.22) that

$$\xi=\sum_{k=1}^{s-1}a_kx_{r_k}. \tag{5.3.23}$$

Since $B_{\Xi,\lambda}$ satisfies condition (b), (5.3.23) implies that

$$\lambda_\xi-\sum_{k=1}^{s-1}a_k\lambda_{r_k}\notin 2\pi i\mathbf{Z}\setminus\{0\}. \tag{5.3.24}$$

From (5.3.17) and (5.3.22) it is easy to see that

$$\mu_{P(\xi)}-\sum_{k=2}^{s-1}a_k\mu_{r_k}=\lambda_\xi-\sum_{k=1}^{s-1}a_k\lambda_{r_k}. \tag{5.3.25}$$

Thus (5.3.25) and (5.3.24) show that $B_{\Xi'',\mu}$ satisfies condition (b) and this finishes the proof.

♠

Note that condition (b) of the theorem above is automatically met if $\lambda\in\mathbf{R}^n$. This leads us to

Corollary 5.3.26. *Let Ξ be an $s \times n$ rational matrix with non-zero columns and rank s. Suppose that $\lambda \in \mathbf{R}^n$ is given. Then $K(B_{\Xi,\lambda})$ has finite dimension provided Ξ has property $\mathcal{E}$.*

Corollary 5.3.26 and Proposition 5.2.5 serve to extend the sufficiency part of Theorem 5.3.3 to rational matrices.

Remark 5.3.27. Condition (a) of Theorem 5.3.11 is not necessary for finite dimensionality unless Ξ is an integer matrix. Take, for example, Ξ to be the matrix given in Remark 4.2.39 and let $\lambda = 0$. Then $K(B_{\Xi,\lambda})$ is trivial, *a fortiori*, finite dimensional. However, Proposition 5.2.6 shows that Ξ does not have property $\mathcal{E}$. Of course, if Ξ is an integer matrix, then the necessity of property $\mathcal{E}$ follows from Theorem 5.3.5 and Proposition 5.2.5.

5.4. An application

Throughout this section, we will assume that Ξ is an $s \times n$ integer matrix and $\lambda = 0$. Suppose that $\mathcal{A}$ is a (perhaps proper) submodule of the $\mathbf{Z}$-module $\mathbf{Z}^s$, and recall (from (1.4.4)) that

$$K_{\mathcal{A}}(B_\Xi) := \left\{ a : \mathcal{A} \to \mathbf{C} : \sum_{k \in \mathcal{A}} a(k) B_\Xi(\cdot - k) = 0 \right\}.$$

If $\mathcal{A}$ is a proper submodule of $\mathbf{Z}^s$, then it is clear from the definition above that any sequence in $K_{\mathcal{A}}(B_\Xi)$ can be canonically extended to one in $K(B_\Xi)$ simply by assigning the value zero at those lattice points in $\mathbf{Z}^s \setminus \mathcal{A}$. So if $K(B_\Xi)$ is finite dimensional, then so too is $K_{\mathcal{A}}(B_\Xi)$. Nonetheless, it is quite conceivable that for a properly chosen submodule $\mathcal{A}$, the space $K_{\mathcal{A}}(B_\Xi)$ has finite dimension while $K(B_\Xi)$ itself may be infinite dimensional. This next result helps identify such submodules. (Compare Corollary 4.2.41.)

Proposition 5.4.1. *Let Ξ be an $s \times n$ integer matrix with rank s and non-zero columns. Let $\mathcal{A}:=A\mathbf{Z}^s$ where A is an invertible $s \times s$ integer matrix. Assume that each $s \times k$ $(k \leq s-1)$ submatrix W of Ξ with full rank can be completed to an $s \times s$ integer matrix $\widetilde{W}$ such that $\mathcal{A} \subseteq \widetilde{W}\mathbf{Z}^s$. Then $K_{\mathcal{A}}(B_\Xi)$ is finite dimensional.*

Proof. Defining $\Xi' := A^{-1}\Xi$, it is not hard to see, using (4.2.20), that $K_{\mathcal{A}}(B_\Xi)$ is finite dimensional if and only if $K(B_{\Xi'})$ is. Evidently Ξ' is a rational matrix and if W' is any $s \times k$ submatrix of Ξ' with full rank, then $W' = A^{-1}W$ for some $s \times k$ submatrix W of Ξ with full rank. By our premise, W can be completed to an $s \times s$ integer matrix $\widetilde{W}$ such that $A\mathbf{Z}^s \subseteq \widetilde{W}\mathbf{Z}^s$, i.e., there is an $s \times s$ integer matrix Q such that $A = \widetilde{W}Q$. Now $\widetilde{W'} = A^{-1}\widetilde{W}$ is a completion of W' and $(\widetilde{W'})^{-1} = Q$, an integer matrix. This shows that Ξ' has property $\mathcal{E}$ and the desired result follows from Corollary 5.3.26.

♠

Observe that if $\mathcal{A} = \mathbf{Z}^s$, then Proposition 5.2.2 shows that the assumption in Proposition 5.4.1 is in fact equivalent to the weak unimodularity of Ξ and thereby to the finite dimensionality of $K_{\mathcal{A}}(B_\Xi)(= K(B_\Xi))$.

Example 5.4.2. Let

$$\Xi := \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 4 \end{pmatrix},$$

$$A := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathcal{A} := A\mathbf{Z}^2.$$

Since Ξ is not weakly unimodular, $K(B_\Xi)$ is infinite dimensional. However, $K_A(B_\Xi)$ has finite dimension. This can be seen from Proposition 5.4.1 or by noting that

$$A^{-1}\Xi =: \Xi' = \begin{pmatrix} 1/2 & 0 & 3 \\ 1/2 & 1 & -1 \end{pmatrix}$$

has property $\mathcal{E}$ (Proposition 5.2.6).

Also note that for the submatrix

$$Y := \begin{pmatrix} 0 & 3 \\ 1 & -1 \end{pmatrix}$$

of Ξ', $|\det Y| = 3$ and so the integer translates of B_Y are linearly dependent ([BH82, Proposition 4] or Theorem 4.2.4). Thus $N(B_Y)$ is non-empty and therefore $N(B_{\Xi'})$ ($\supseteq N(B_Y)$) is non-void as well. This shows that $K(B_{A^{-1}\Xi})$ and hence $K_A(B_\Xi)$ is non-trivial.

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